• Higher Math Literacy Test Prescriptives •

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• Standard to Slope–Intercept Form •

Sometimes we want to convert between forms in order to more easily understand the components of an equation. For example, we can more easily identify the slope and ***y*–INTERCEPT** of an equation written in **SLOPE–INTERCEPT FORM** and more easily identify the ***x*–INTERCEPT** of an equation written in **STANDARD FORM**.

We can rewrite the equation of a line from **STANDARD FORM** to **SLOPE–INTERCEPT FORM** by isolating the ***y*** variable.

EXAMPLE: Write $2x - 8y = 16$ in **SLOPE–INTERCEPT FORM**.

Steps to solve:

STEP 1:
Move the term containing ***x*** to the right side of the equation.

$$\begin{aligned} 2x - 8y &= 16 \\ -2x \qquad &\quad -2x \\ \hline -8y &= -2x + 16 \end{aligned}$$

STEP 2:
Divide each term by the coefficient of ***y***.

$$\frac{-8y}{-8} = \frac{-2x}{-8} + \frac{16}{-8}$$

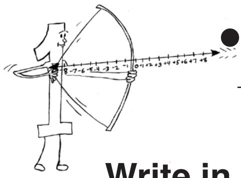
$$y = \frac{1}{4}x - 2$$

Try these: **Write in SLOPE–INTERCEPT FORM ($y = mx + b$).**

1) $2x - 3y = 15$

2) $4x + 3y = -9$

____________________ ____________________



• Standard to Slope–Intercept Form •

Write in SLOPE–INTERCEPT FORM ($y = mx + b$).

1) $2x + 9y = 9$

____

2) $14x + 7y = 2$

____

3) $12x - 9y = {}^{-}16$

____

4) $5x - 2y = {}^{-}8$

____

5) $x - y = {}^{-}6$

____

6) $15x - 4y = 12$

____

7) $7x + 6y = {}^{-}8$

____

8) $3x - 2y = 9$

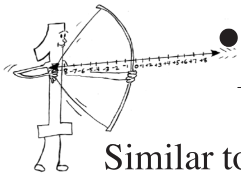
____

9) $14x + 17y = {}^{-}34$

____

10) $9x - 2y = 13$

____



• Slope–Intercept to Standard Form •

Similar to rewriting the equation of a line from STANDARD to SLOPE–INTERCEPT FORM, we can rewrite the equation of a line from SLOPE–INTERCEPT to STANDARD FORM. We do this by converting all the numbers into integers and then rearranging the equation.

EXAMPLE: Write $y = \frac{1}{4}x - \frac{1}{8}$ in STANDARD FORM.

Steps to solve:

STEP 1:
Because there are fractions in the equation, multiply the terms of the equation by the **LCM** of the denominators.

LCM of 4 and 8: 8

$$8 \bullet y = 8\left(\frac{1}{4}x - \frac{1}{8}\right)$$

$$8y = 2x - 1$$

STEP 2:
Move the term containing x to the left side of the equation.

NOTE: We write the term containing x before the term containing y when converting to STANDARD FORM.

$$\begin{array}{rcl} 8y & = & 2x - 1 \\ -2x & & -2x \\ \hline {}^{-}2x + 8y & = & 0 - 1 \end{array}$$

$${}^{-}2x + 8y = {}^{-}1$$

STEP 3:
If the coefficient of x is negative, multiply the terms of the equation by ${}^{-}1$.

$${}^{-}2x + 8y = {}^{-}1$$

$${}^{-}1({}^{-}2x + 8y) = {}^{-}1 \bullet {}^{-}1$$

$$2x - 8y = 1$$

***Try these*:** **Write in STANDARD FORM ($ax + by = c$).**

1) $y = 2x + 3$

2) $y = \frac{1}{4}x - 2$

______________________ ______________________

• Slope–Intercept to Standard Form • ☆

Write in STANDARD FORM ($ax + by = c$).

1) $y = -\frac{1}{2}x - 19$

____

2) $y = \frac{7}{24}x + 2$

____

3) $y = \frac{3}{8}x - \frac{1}{10}$

____

4) $y = {}^{-}18x - 25$

____

5) $y = -\frac{4}{7}x + 3$

____

6) $y = \frac{13}{22}x + \frac{1}{11}$

____

7) $y = 4x + \frac{3}{7}$

____

8) $y = 2x - 4.5$

____

9) $y = -\frac{8}{9}x - \frac{2}{3}$

____

10) $y = x$

____

• Changing Forms •

☆

Write in STANDARD FORM.

1) $y = -\frac{1}{4}x + 3$

2) $y = -\frac{1}{5}x - \frac{1}{3}$

____

____

3) Create an equation written in **SLOPE–INTERCEPT FORM.**

____

Now, write your equation in **STANDARD FORM.**

____

Write in SLOPE–INTERCEPT FORM.

4) $16x + 2y = {}^{-}8$

5) $4x - 5y = 9$

____

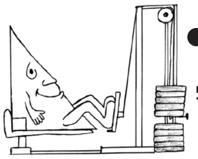
____

6) Create an equation written in **STANDARD FORM.**

____

Now, write your equation in **SLOPE–INTERCEPT FORM.**

____



• Mastery Check: Changing Forms •

Write in SLOPE–INTERCEPT FORM.

1) $5x - 2y = 6$

2) $10x - 5y = 9$

3) $17x + 6y = 12$

4) $3x - 9y = 20$

Write in STANDARD FORM.

5) $y = \frac{2}{3}x + 4$

6) $y = \frac{1}{9}x - 4$

7) $y = {}^{-}3x + 11$

8) $y = -\frac{1}{5}x - \frac{1}{8}$

Challenge: Write in SLOPE–INTERCEPT FORM.

9) $\frac{a}{b}x + \frac{c}{b}y = d$

• Independent and Dependent Variables •

Direct Teaching

Every function has a set of input values and a set of output values.

The input value is known as the **INDEPENDENT VARIABLE**.

The output value is known as the **DEPENDENT VARIABLE**. This variable *depends* on the input value. Typically, the dependent variable is isolated on one side of the equation.

Let's look at an example to determine the independent and dependent variables.

EXAMPLE: Determine the independent and dependent variables in the equation $y = \frac{1}{3}x + 1$.

Input (x)	Output (y)
0	1
3	2
6	3
9	4

In the equation, we have y isolated on one side and all other terms on the other side. When we substitute an input value for x, we get an output value for y.

This means that x is the input value and y is the output value. In other words, the y value *depends* on the x value.

So, x is the **INDEPENDENT VARIABLE** and y is the **DEPENDENT VARIABLE**.

Try these: **Determine the independent and dependent variables in the equation.**

1) $y = 5x - 3$

Independent variable: _____

Dependent variable: _____

2) $x = -\frac{1}{2}y + 5$

Independent variable: _____

Dependent variable: _____

3) $^{-}3x = y$

Independent variable: _____

Dependent variable: _____

4) $\frac{4}{5}a - 2 = b$

Independent variable: _____

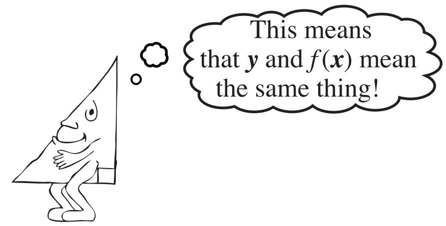
Dependent variable: _____

• Function Notation •

An equation is in function notation when the *dependent variable* is written as $f(\boldsymbol{x})$. The letter outside the parentheses, usually f, stands for *function* and the variable inside the parentheses is the independent variable.

Since most linear equations are written in the form $\boldsymbol{y} = m\boldsymbol{x} + b$, we replace the $\boldsymbol{y}$ with $f(\boldsymbol{x})$ to write the linear equation as a function.

$$\boldsymbol{y} = f(\boldsymbol{x})$$



Example: Write the equation $\boldsymbol{y} = \frac{2}{3}\boldsymbol{x} - 1$ in function notation.

Since the $\boldsymbol{y}$ value *depends* on the $\boldsymbol{x}$ value, $\boldsymbol{y}$ is the dependent variable and $\boldsymbol{x}$ is the independent variable.

So, $\boldsymbol{y} = \frac{2}{3}\boldsymbol{x} - 1$ written in function notation is $f(\boldsymbol{x}) = \frac{2}{3}\boldsymbol{x} - 1$.

Try these: **Determine the independent and dependent variables in the equation. Then write the equations in function notation.**

1) $\boldsymbol{y} = 5\boldsymbol{x} - 3$

Independent variable: _____

Dependent variable: _____

$f($___$) =$ ______________

2) $2\boldsymbol{b} + 4\boldsymbol{a} = 6$ in terms of $\boldsymbol{a}$

Independent variable: _____

Dependent variable: _____

$f($___$) =$ ______________

3) $\boldsymbol{y} = 3\boldsymbol{x}^2 + 5\boldsymbol{x} - 2$

Independent variable: _____

Dependent variable: _____

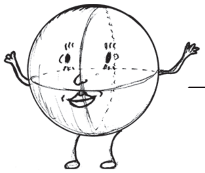
$g($___$) =$ ______________

4) $\boldsymbol{x} = \boldsymbol{y} - 8$

Independent variable: _____

Dependent variable: _____

$h($___$) =$ ______________



• Function Notation •

☆

Write the equations in function notation.

1) $y = -\frac{3}{5}x + 1$

$f(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

2) $3l + 7 = w$

$f(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

3) $x = {}^{-}4y - 2$

$f(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

4) $a = 10b$

$f(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

5) $j + 2 = 3k + 5$ in terms of k

$g(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

6) $2a - 3b = 6$ in terms of b

$f(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

7) $y = \frac{1}{2}x + 4$

$h(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

8) $y = 10z$

$v(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

9) $\frac{1}{2}p - 2 = 3q - 3$ in terms of p

$g(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

10) $y = 8$

$h(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

• Evaluating a Function •

We can evaluate a function for a given value. Whatever value replaces the independent variable in function notation is substituted everywhere the variable is in the function.

So, if $f(\boldsymbol{x}) = \boldsymbol{x}^2 + 3\boldsymbol{x} - 10$, then:

If we evaluate when $\boldsymbol{x} = 11$, then $f(11) = (11)^2 + 3(11) - 10 = 144$.

If we evaluate when $\boldsymbol{x} = 0$, then $f(0) = (0)^2 + 3(0) - 10 = {}^{-}10$.

If we evaluate when $\boldsymbol{x} =$ ☺, then $f($☺$) = ($☺$)^2 + 3($☺$) - 10 =$ ☺$^2 + 3$☺ $- 10$.

If the value substituted is a number, we can get a numeric value for the output.

Example 1: If $f(\boldsymbol{x}) = 4\boldsymbol{x} + 17$, then evaluate $f(5)$.

If we evaluate when $\boldsymbol{x} = 5$, then $f(5) = 4(5) + 17 = 20 + 17 = 37$.

So, $f(5) = 37$.

Example 2: If $f(\boldsymbol{h}) = 12\boldsymbol{h}^2 - 6\boldsymbol{h} + 1$, then evaluate $f({}^{-}1)$.

If we evaluate when $\boldsymbol{h} = {}^{-}1$, then $f({}^{-}1) = 12({}^{-}1)^2 - 6({}^{-}1) + 1 = 12 + 6 + 1 = 19$.

So, $f({}^{-}1) = 19$.

Try these: **Evaluate the function.**

1) $f(\boldsymbol{x}) = 10\boldsymbol{x} + 3$
Find $f(0)$.

$f(0) =$ ______________

2) $f(\boldsymbol{w}) = 5\boldsymbol{w}^3 - \boldsymbol{w}$
Find $f(2)$.

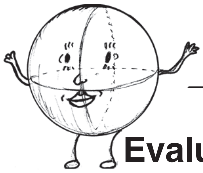
$f(2) =$ ______________

3) $g(\boldsymbol{x}) = \boldsymbol{x}^2 - 12\boldsymbol{x} - 24$
Find $g(1)$.

$g(1) =$ ______________

4) $f(\boldsymbol{k}) = 12\boldsymbol{k}^3$
Find $f($☼$)$.

$f($☼$) =$ ______________



• Evaluating a Function •

Evaluate the function.

1) $f(x) = {}^{-}4x - 2$
Find $f(2)$.

$f(2) =$ ________________

2) $f(j) = 5j + 12$
Find $f(2)$.

$f(2) =$ ________________

3) $h(x) = x^3 + 3x - 2$
Find $h({}^{-}1)$.

$h({}^{-}1) =$ ________________

4) $f(x) = 4x^2 + 3$
Find $f(c)$.

$f(c) =$ ________________

5) $g(x) = x^4 + 5x^2 + 10$

$g(0) =$ ________________

6) $f(k) = 3k^2$

$f({}^{-}2) =$ ________________

7) $g(u) = u^2 - 12u - 24$

g(☺) = ________________

8) $p(x) = (x - 3)(x + 5)$

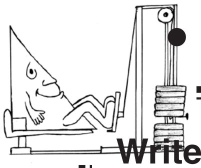
$p(3) =$ ________________

9) $v(t) = t^2 + 5t - 50$

$v(t + 1) =$ ________________

10) $f(x) = 12$

$f(a) =$ ________________



• Mastery Check: Function Notation • ☆

Write the equations in function notation.

1) $y = \frac{3}{4}x + 6$

$f($___$) =$ _______________

2) $2j + 2 = k$

$f($___$) =$ _______________

3) $\frac{1}{2}a = 4b - 8$ in terms of a

$f($___$) =$ _______________

4) $3u + 6v = {}^{-}12$ in terms of v

$f($___$) =$ _______________

Evaluate the function.

5) $f(x) = 3x^2 - 4x + 5$

$f(2) =$ _______________

6) $f(v) = 2v^3$

$f(3) =$ _______________

7) $h(x) = x^2 + 5x + 6$

$h({}^{-}2) =$ _______________

8) $g(x) = x^2$

$g(x + h) =$ _______________

Challenge:

9) $h(n) = 10x - 2$

$h(3) =$ _______________

10) $f(x) = (x - 3)(z - 1)(x + 2)(z + 3)$

$f(4) =$ _______________

• Writing Algebraic Equations •

The first step in solving any algebraic word problem is to use the information in the scenario to write at least one algebraic equation. When we are given a statement that compares two unknown values, there are two equivalent equations we can write based on the given information.

EXAMPLE 1: Armand has 5 fewer books than Bernard. Write an equation that compares the number of books each has:

In terms of Bernard's books: $A = B - 5$

In terms of Armand's books: $B = A + 5$

EXAMPLE 2: Armand has 5 times as many pencils as Bernard. Write an equation that compares the number of pencils each has:

In terms of Bernard's pencils: $A = 5B$

In terms of Armand's pencils: $B = \frac{A}{5}$

Try these: **Write two algebraic equations that represent the comparison in each statement below.**

1) Emilie has 8 more hats than Francine.

E = ____________________

F = ____________________

2) Emilie has 3 fewer mittens than Francine.

E = ____________________

F = ____________________

3) Emilie has 6 times as many scarves as Francine.

E = ____________________

F = ____________________

4) Emilie has half as many jackets as Francine.

E = ____________________

F = ____________________

• Writing Algebraic Equations •

Write an algebraic equation that represents each comparison in terms of the weight of the puppy. Use the variable *p* for the weight of the puppy.

1) A puppy weighs 10 pounds more than a kitten.

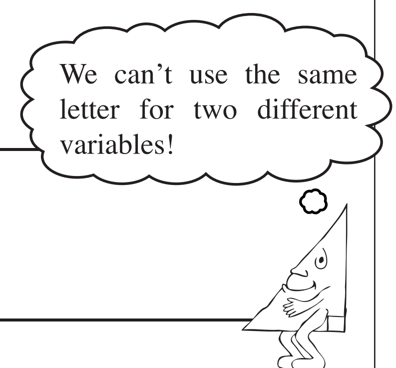
____

2) The puppy weighs twice as much as a rabbit.

____

3) A koi weighs a tenth the weight of the puppy.

____



4) A pony weighs 200 pounds more than the puppy.

____

Write two algebraic equations that represent the comparison in each statement below.

5) Dishwasher detergent costs a third as much as paper towels.

____

____

6) A box of thumbtacks costs $4.00 less than a box of markers.

____

____

7) A carton of eggs costs $1 more than a gallon of milk.

____

____

8) A magazine costs 4 times as much as a packet of chocolates.

____

____

• Writing Systems of Equations • ☆

Write two equations based on the information given. Make sure that each equation is set equal to the same unknown. Do not solve.

1) There are a total of 24 eggs in a basket. There are twice as many brown eggs as there are white eggs.

① $b = 24 - w$
② $b = 2w$

2) Harry has 150% the number of posters that Ines has. Ines has 2 fewer posters than Harry.

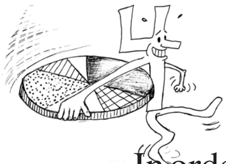
①
②

3) Every day, a store sells half as many apples as bananas. On Tuesday, they sold 58 more bananas than apples.

①
②

4) Justin has 5 times as many comic books as Kari. Kari has 40 fewer comic books than Justin.

①
②



• Systems of Equations •

In order to solve for the values of two unknown constants, we need enough information to write two different equations. Once we write our system of equations, we can solve for both unknown values.

Example: Maria has two thirds the amount of homework that Nancy does. Maria has 3 pages of homework fewer than Nancy. How many pages of homework does each person have?

Steps to solve:

Step 1:
Write two equations based on the information given. Make sure that each equation is set equal to the same unknown.

$$\begin{cases} ① \ \boldsymbol{M} = \frac{2}{3}\boldsymbol{N} \\ ② \ \boldsymbol{M} = \boldsymbol{N} - 3 \end{cases}$$

Step 2:
Since both expressions in terms of ***N*** are equal to ***M***, substitute the equivalent expression for ***M*** from equation ① in place of ***M*** in equation ②.

$$\frac{2}{3}N = N - 3$$

$$2N = 3N - 9$$

$$N = 9$$

Step 3:
Now that we have the value of one variable, we can substitute it into an equation from the system to solve for the other.

$$\boldsymbol{M} = \boldsymbol{N} - 3$$

$$\boldsymbol{M} = 9 - 3$$

$$\boldsymbol{M} = 6$$

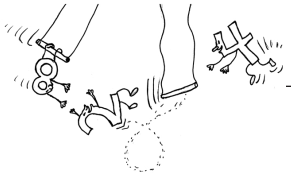
Nancy has 9 pages of homework and Maria has 6.

***Try this*: Use a system of two equations to solve.**

1) Opal has half as many marbles as Peter. Peter has 23 more than Opal does. How many marbles does each have?

$$\begin{cases} ① \\ ② \end{cases}$$

Opal: __________ Peter:__________



• Systems of Equations •

Complete the exercises by using a system of equations.

1) Randall can do 4 times as many jumping jacks as Steven. They start doing jumping jacks at the same time, but Randall does 18 more jumping jacks after Steven stops. How many jumping jacks can each of them do?

①

②

Randal: __________ Steven: __________

2) A television costs 125% the price of a sound system. The television costs $24 more than the sound system. How much does each cost?

①

②

Television: __________ Sound System: __________

• Systems of Equations Problems • ☆

If we have as many equations as we have unknowns (and no two equations are equivalent), then we have enough information to solve for the value of any or all of the unknowns. One way to do this is to find one value all the equations have in common, write the equations in terms of that value, and then use substitution.

Example: Bernard has \$7 less than Armand. Claudette has twice as much as Armand. Together they have \$53. How much money does each have?

Steps to solve:

Step 1:
Write a system of equations. The equations that represent the comparisons should be in terms of the value they have in common.

$$\begin{cases} ① \; B = A - 7 \\ ② \; C = 2A \\ ③ \; A + B + C = 53 \end{cases}$$

Step 2:
Substitute the expressions from equations ① and ② into equation ③ so that you have one equation with one variable. Solve.

$$A + (A - 7) + (2A) = 53$$
$$4A = 60$$
$$A = 15$$

Armand has \$15.

Step 3:
Plug the value you found using equation ③ into equations ① and ② to solve for the values of the other unknowns.

$$B = 15 - 7 \qquad C = 2 \cdot 15$$
$$B = 8 \qquad C = 30$$

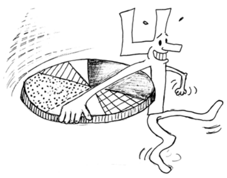
Bernard has \$8 and Claudette has \$30.

Try this: **Identify the term that the comparisons have in common, then write and solve a system of equations based on the scenario.**

1) Michelle has 10 fewer raffle tickets than Noah. Ollie has 3 times as many as Michelle. Together they have 60 raffle tickets. How many does Ollie have?

①
②
③

O = ___________



• Systems of Equations Problems •

Complete the exercises by identifying the term all comparisons have in common and using a system of equations.

1) Gerard has $20 more than Harris. Isabelle has half as much as Gerard. Together they have $105. How much does each person have?

①

②

③

Gerard: __________ Harris: __________ Isabelle: __________

Check your answer: _______ + _______ + _______ = _______

2) A kitten weighs 1 pound and 8 ounces more than a bunny. A puppy weighs $2\frac{1}{2}$ times as much as the kitten. Altogether, they weigh 14 pounds and 4 ounces. How much does each pet weigh?

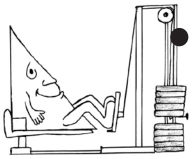
①

②

③

Kitten: __________ Bunny: __________ Puppy: __________

Check your answer: _______ + _______ + _______ = _______



• Mastery Check: Systems Problems •

Complete the exercises by using a system of equations.

1) A history book weighs 24 ounces more than a math book. The math book is 75% the weight of the history book. How much does each book weigh?

History: __________ Math: __________

2) Deandra has $15 more than Elias. Frank has 3 times as much as Deandra. Together they have $185. How much does each person have?

Deandra: __________ Elias: __________ Frank: __________

Challenge:

3) Noelle has 15 more cookies than Owen. Peter has 3 times as many cookies as Noelle. Quinn has half as many cookies as Noelle. Together they have 95 cookies. How many does each person have?

Noelle: ________ Owen: ________ Peter: ________ Quinn: ________

• Solving a System of Linear Equations by Elimination • ☆

Direct Teaching

One method of solving a linear system is **elimination**. In the elimination method, the equations are added together or subtracted to eliminate a variable and obtain a new equation in the remaining variable. After solving this equation for the variable, it can be substituted into one of the original equations to solve for the other variable.

In the linear system below, the coefficients of y, 3 and $^-3$, are **opposite**, as their sum is zero ($3 + {}^-3 = 0$). When the coefficients of a variable are **opposite**, **add** the equations together to eliminate the variable.

Example: Solve the linear system by elimination. Check your answer.

$$\begin{cases} ① \ 4x + 3y = 15 \\ ② \ 5x - 3y = 12 \end{cases}$$

Steps to Solve

Step 1:
Since the coefficients of y are opposite, add equations ① and ② together to eliminate y.

$$\begin{array}{rl} ① & 4x + 3y = 15 \\ + \ ② & (5x - 3y = 12) \\ \hline & 9x \qquad = 27 \end{array}$$

Step 2:
Solve the remaining equation for x.

$$9x = 27$$
$$x = \mathbf{3}$$

Step 3:
Substitute the value of x into equation ① or ② to find the value of y and complete the solution.

$$① \ 4(\mathbf{3}) + 3y = 15$$
$$12 + 3y = 15$$
$$y = \mathbf{1}$$

So, $(x, y) = (3, 1)$.

Verify the solution by substituting the value of x and y into each original equation.

$① \ 4(3) + 3(1) = 12 + 3 \overset{\checkmark}{=} 15$ $\qquad$ $② \ 5(3) - 3(1) = 15 - 3 \overset{\checkmark}{=} 12$

***Try this*: Solve the linear system by elimination. Check your answer.**

1) $\begin{cases} ① \ 2x - 5y = {}^-11 \\ ② \ {}^-2x + y = 7 \end{cases}$

$(x, y) =$ ___________

Elimination Method – Same Coefficients

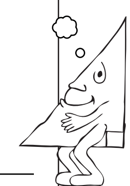
The elimination method can also be used to solve a linear system by eliminating a variable that has the **same coefficient** in each equation.

When the coefficient of the variable is the **same** in each equation, **subtract** one equation from the other to eliminate the variable.

EXAMPLE: Solve the linear system by elimination. Check your answer.

$$\begin{cases} ① \ 6x + 9y = 3 \\ ② \ 6x - 2y = 36 \end{cases}$$

Remember, subtracting $^{-}2y$ is the same as adding $2y$.



Steps to Solve

STEP 1:
Since x has the same coefficient in each equation, subtract equation ② from ① to eliminate x.

$$\begin{array}{rl} ① & 6x + 9y = 3 \\ - \ ② & (6x - 2y = 36) \\ \hline & 11y = {}^{-}33 \end{array}$$

STEP 2:
Solve the remaining equation for y.

$$11y = {}^{-}33$$
$$y = {}^{-}3$$

STEP 3:
Substitute the value of y into equation ① or ② to find the value of x and complete the solution.

$$① \ 6x + 9({}^{-}3) = 3$$
$$6x - 27 = 3$$
$$x = 5$$

So, $(x, y) = (5, {}^{-}3)$.

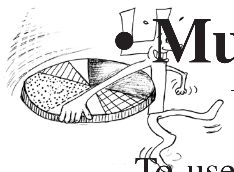
Verify the solution by substituting the value of x and y into each original equation.

$① \ 6(5) + 9({}^{-}3) = 30 - 27 \stackrel{\checkmark}{=} 3$ $\quad$ $② \ 6(5) - 2({}^{-}3) = 30 + 6 \stackrel{\checkmark}{=} 36$

Try this: **Solve the linear system by elimination. Check your answer.**

1) $\begin{cases} ① \ 2x + 5y = 2 \\ ② \ 4x + 5y = {}^{-}6 \end{cases}$

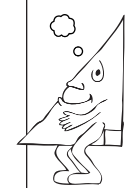
$(x, y) =$ ___________



• Multiplying an Equation Before Elimination • ☆

To use the elimination method, it is sometimes necessary to first **multiply** one or more equations to have variables with the **same** or **opposite** coefficients in the linear system.

You can also use **elimination** by multiplying equation ① by $^{-}3$ and adding the equations to eliminate $\boldsymbol{x}$, if you prefer.



EXAMPLE: Solve the linear system by elimination.

$\begin{cases} ① \ 6x + 7y = 9 \\ ② \ 2x + 5y = 11 \end{cases}$	$3(2x + 5y) = 3(11)$ $6x + 15y = 33$	① $6x + 7y = 9$ $6x + 15y = 33$
A linear system that can be solved using the elimination method.	*Multiply both sides of equation ② by 3 to change the coefficient of* $\boldsymbol{x}$ *from 2 to 6.*	*With the same coefficient of* $\boldsymbol{x}$ *in both equations,* $\boldsymbol{x}$ *can be eliminated by subtraction.*

***Try these*: Solve the linear system by elimination. Check your answer.**

1) $\begin{cases} ① \ ^{-}4x + 3y = 1 \\ ② \ 5x - 6y = {^{-}8} \end{cases}$

$(x, y) =$ __________

2) $\begin{cases} ① \ 2x + 5y = 18 \\ ② \ 4x + y = 0 \end{cases}$

$(x, y) =$ __________

3) $\begin{cases} ① \ 4x - 2y = {^{-}6} \\ ② \ ^{-}8x + 3y = 7 \end{cases}$

$(x, y) =$ __________

• Multiplying Both Equations Before Elimination •

It can be useful to first rewrite **both** equations using multiplication before using the elimination method to solve a linear system.

Example: Solve the linear system by elimination. Check your answer.

$$\begin{cases} ① \ 3x + 4y = 11 \\ ② \ 4x + 5y = 14 \end{cases}$$

Steps to Solve

Step 1:
Multiply each equation so that the coefficients of x are both 12, the LCM of 3 and 4.

$$4(3x + 4y) = 4(11)$$
$$3(4x + 5y) = 3(14)$$

Step 2:
Now that x has the same coefficient in both equations, subtract the new equation ② from the new equation ① to eliminate x and solve for y.

$$\begin{array}{r} 12x + 16y = 44 \\ -\ (12x + 15y = 42) \\ \hline y = \mathbf{2} \end{array}$$

Step 3:
Substitute the value of y into equation ① or ② to find the value of x and complete the solution.

$$3x + 4(\mathbf{2}) = 11$$
$$3x + 8 = 11$$
$$x = \mathbf{1}$$

So, $(x, y) = (1, 2)$.

Verify the solution by substituting the value of x and y into each original equation.

① $3(\mathbf{1}) + 4(\mathbf{2}) = 3 + 8 \overset{\checkmark}{=} 11$ ② $4(\mathbf{1}) + 5(\mathbf{2}) = 4 + 10 \overset{\checkmark}{=} 14$

Try this: **Solve the linear system by elimination. Check your answer.**

1) $\begin{cases} ① \ 2x - 3y = {}^{-}6 \\ ② \ 3x - 4y = {}^{-}7 \end{cases}$

$(x, y) =$ ____________

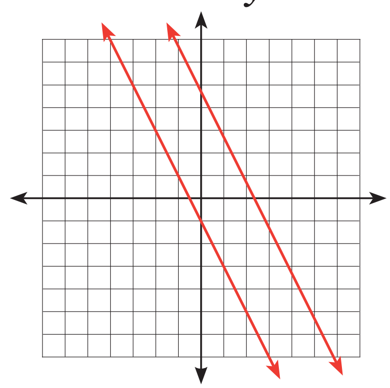
• Inconsistent System – No Solution Exists • ☆

Observe the result when attempting to solve the following linear system:

$$\begin{cases} ① \ 4x + 2y = {}^{-}2 \\ ② \ 2x + y = 5 \end{cases} \qquad \begin{aligned} 2(2x + y) &= 2(5) \\ 4x + 2y &= 10 \end{aligned}$$

$$\begin{array}{r} ① \ 4x + 2y = {}^{-}2 \\ -\ \ (4x + 2y = 10) \\ \hline 0 \overset{?}{=} {}^{-}12 \end{array} \qquad 0 \overset{\times}{=} {}^{-}12$$

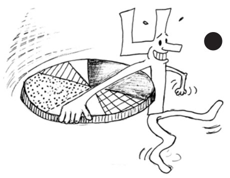
The final equation is **impossible**. When the linear system is graphed, we can see two **parallel** lines that do not intersect. Therefore, there is **no solution**.



A linear system that has no solution is an **INCONSISTENT SYSTEM**. When attempting to use the elimination method to solve an **INCONSISTENT SYSTEM**, an impossible equation will result.

Try this: **Use the elimination method to solve the linear system. If an impossible equation results, write the impossible equation and "no solutions."**

1) $\begin{cases} ① \ 3x - 2y = 1 \\ ② \ 6x - 4y = 8 \end{cases}$

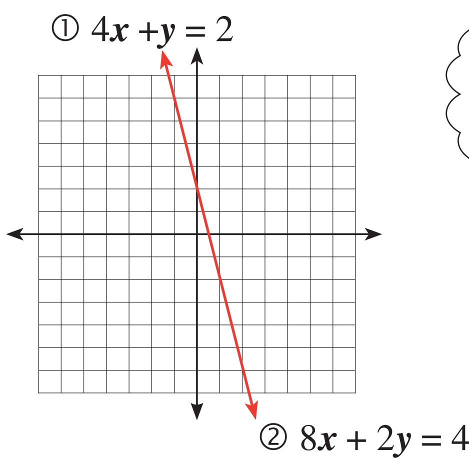


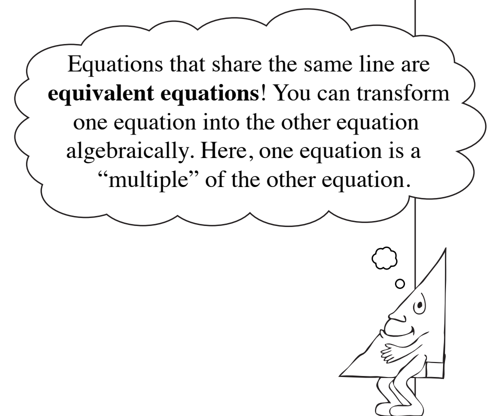
• Dependent System – Infinite Solutions • ☆

Observe the result when attempting to solve the following linear system:

$$\begin{cases} ① \; 4x + y = 2 \\ ② \; 8x + 2y = 4 \end{cases} \qquad \begin{aligned} 2(4x + y) &= 2(2) \\ 8x + 2y &= 4 \end{aligned} \qquad \begin{aligned} 8x + 2y &= 4 \\ -\;(8x + 2y &= 4) \\ \hline 0 &\stackrel{?}{=} 0 \end{aligned} \qquad 0 = 0$$

In the final equation, the variables have been eliminated and the equation is valid. When the linear system is graphed, we can see that each equation is represented by the **same** line. Therefore, there are **infinite solutions** – every point on the line is a solution to both equations.





A linear system that has infinite solutions is a **DEPENDENT SYSTEM**. When attempting to use the elimination method to solve a **DEPENDENT SYSTEM**, a valid equation without variables will result.

Try this: **Use the elimination method to solve the linear system. If a true equation results without any variables, write the equation and “infinite solutions.”**

1) $\begin{cases} ① \; 3x - 4y = {}^-1 \\ ② \; 6x - 8y = {}^-2 \end{cases}$

____

Solve the linear system by elimination. Check your answer. If there are no solutions or infinite solutions, write "no/infinite solutions."

1) $\begin{cases} ① \ 2x - 5y = 1 \\ ② \ 3x + 4y = 13 \end{cases}$

____

2) $\begin{cases} ① \ 2x + 4y = 16 \\ ② \ x + 2y = 5 \end{cases}$

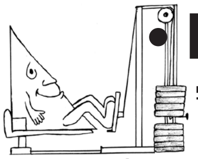
____

3) $\begin{cases} ① \ \frac{1}{2}x - \frac{3}{4}y = 0 \\ ② \ \frac{1}{2}x + \frac{1}{4}y = 4 \end{cases}$

____

4) $\begin{cases} ① \ 5x - 6y = 14 \\ ② \ {}^{-}5x + 9y = {}^{-}11 \end{cases}$

____



• Mastery Check: Elimination Method •

☆

Solve the linear system by elimination. Check your answer. If there are no solutions or infinite solutions, write "no/infinite solutions."

1) $\begin{cases} ① \ 2x + 3y = 8 \\ ② \ 5x - 6y = {}^{-}7 \end{cases}$

____

2) $\begin{cases} ① \ x + 4y = 9 \\ ② \ 2x + 8y = 5 \end{cases}$

____

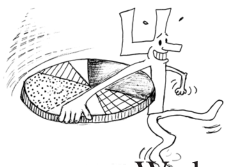
3) $\begin{cases} ① \ 5x - 7y = {}^{-}11 \\ ② \ {}^{-}5x + 3y = {}^{-}1 \end{cases}$

____

Challenge:

4) $\begin{cases} ① \ x - y + z = 9 \\ ② \ x + y - z = 3 \\ ③ \ {}^{-}x - y - z = {}^{-}7 \end{cases}$

____



• Factoring Quadratic Polynomials • ☆

Direct Teaching

We have seen how to factor a QUADRATIC POLYNOMIAL where $a = 1$, so let's look at an example where $a > 1$.

EXAMPLE: Factor $2x^2 + 5x - 12$.

Steps to Solve:

STEP 1: First, identify the values of a, b, and c. $a = 2, b = 5, c = {}^{-}12$

STEP 2: Determine which two integers multiply to make $a \bullet c$ ($2 \bullet {}^{-}12 = {}^{-}\mathbf{24}$) and *add* to make b (**5**).

$${}^{-}3 \bullet 8 = {}^{-}\mathbf{24}$$
$${}^{-}3 + 8 = \mathbf{5}$$

STEP 3: "Split" the middle term into two terms whose coefficients are the numbers found in **STEP 2**.

$$2x^2 + 5x - 12$$
$$= 2x^2 + {}^{-}3x + 8x - 12$$

STEP 4: Finally, factor the resulting polynomial by grouping.

$$(2x^2 - 3x) + (8x - 12)$$
$$= x(2x - 3) + 4(2x - 3)$$
$$= (2x - 3)(x + 4)$$

So, $2x^2 + 5x - 12$ factored is $(2x - 3)(x + 4)$.

Try these: **Factor the following polynomials.**

1) $3x^2 + 7x - 6$

$a \bullet c =$ <u>$^{-}18$</u> $b =$ <u>7</u>

$= 3x^2 +$ ____ $+$ ____ $- 6$

$=$ ____________________

2) $2y^2 - 5y - 12$

$a \bullet c =$ ____ $b =$ ____

$= 2y^2 +$ ____ $+$ ____ $- 12$

$=$ ____________________

• Factoring Quadratic Polynomials •

Factor the following polynomials.

1) $5x^2 + 6x + 1$

$a \bullet c =$ _____ $b =$ _____

$= 5x^2 +$ _____ $+$ _____ $+ 1$

$=$ ____________________________

2) $2w^2 - 7w + 5$

$a \bullet c =$ _____ $b =$ _____

$= 2w^2 +$ _____ $+$ _____ $+ 5$

$=$ ____________________________

3) $2m^2 + m - 3$

$=$ ____________________________

4) $6x^2 + 13x + 5$

$=$ ____________________________

5) $3z^2 - 20z - 7$

$=$ ____________________________

6) $2x^2 + 23x + 30$

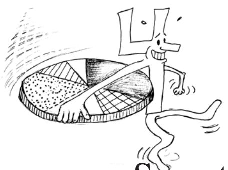
$=$ ____________________________

7) $6x^2 - 13x + 5$

$=$ ____________________________

8) $4y^2 + 7y - 15$

$=$ ____________________________



• Factoring Quadratic Polynomials •

Sometimes the value of a in a **QUADRATIC POLYNOMIAL** is negative. In this case, we factor out a $^{-}1$ to make the value of a positive and then we factor as usual.

EXAMPLE: Factor $^{-}x^2 - 4x + 21$.

$$^{-}x^2 - 4x + 21$$
$$= {^{-}1}(x^2 + 4x - 21)$$
$$= {^{-}(x + 7)(x - 3)}$$

So, $^{-}x^2 - 4x + 21$ factored is $^{-}(x + 7)(x - 3)$.

Try these: **First factor out a $^{-}1$, then factor completely.**

1) $^{-}y^2 - 10y + 11$

$= {^{-}1}$ (____________________)

= ________________________

2) $^{-}x^2 + 10x - 25$

$= {^{-}1}$ (____________________)

= ________________________

3) $^{-}2x^2 + 3x + 9$

$= {^{-}1}$ (____________________)

= ________________________

4) $^{-}5x^2 - 11x + 12$

$= {^{-}1}$ (____________________)

= ________________________

• Factoring Quadratic Polynomials •

Sometimes in a **QUADRATIC POLYNOMIAL**, all the terms have one or more factors in common. In this case, we factor out the **GREATEST COMMON FACTOR** (**GCF**) of the terms and then factor as usual.

EXAMPLE: Factor $2x^2 - 12x + 16$.

$$\begin{aligned} &2x^2 - 12x + 16 \\ =\ &2(x^2 - 6x + 8) \\ =\ &2(x - 4)(x - 2) \end{aligned}$$

So, $2x^2 - 12x + 16$ factored is $2(x - 4)(x - 2)$.

Try these: **First factor out the GCF of the terms, then factor completely.**

1) $5z^2 - 10z - 15$

= ____ (______________________)
GCF

= ___________________________

2) $3x^2 + 21x + 36$

= ____ (______________________)
GCF

= ___________________________

3) $30x^2 + 200x - 70$

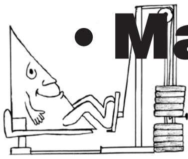
= ____ (______________________)
GCF

= ___________________________

4) $4n^2 - 6n - 18$

= ____ (______________________)
GCF

= ___________________________



• Mastery Check: Factoring Quadratic Polynomials • ☆

Factor the following polynomials.

1) $^-x^2 - 2x + 3$

= ______________________

2) $4y^2 + 6y + 2$

= ______________________

3) $3z^2 + 8z - 3$

= ______________________

4) $3b^2 + 10b - 8$

= ______________________

5) $^-6n^2 + 17n - 5$

= ______________________

6) $3x^2 - 21x - 24$

= ______________________

Challenge:

7) $^-2x^4 - 3x^3 + 9x^2$ = ______________________

• Writing Quadratic Equations in Standard Form • ☆

Consider the following quadratic equation: $x^2 + 5x = {}^{-}6$.

When solving a quadratic equation that is ***not*** in **standard form** ($ax^2 + bx + c = 0$, where $a \neq 0$), it is useful to rewrite the quadratic equation in **standard form** first.

Once the quadratic equation is in standard form, it can be solved by factoring or the quadratic formula.

EXAMPLE: **Rewrite the quadratic equation in standard form. Then, solve. Check your answer.**

$2k^2 - 3 = {}^{-}k$

A quadratic equation, not in standard form.

$2k^2 + k - 3 = 0$

*Add **k** to each side of the equation to rewrite the quadratic equation in standard form.*

$(2k + 3)(k - 1) = 0$

$k = -\frac{3}{2}$ or $k = 1$

Factor the quadratic equation and solve.

Check the solutions: $2(-\frac{3}{2})^2 - 3 = \frac{9}{2} - \frac{6}{2} = \frac{3}{2} \overset{\checkmark}{=} -(-\frac{3}{2})$ $\quad 2(1)^2 - 3 = 2 - 3 \overset{\checkmark}{=} {}^{-}1$

Try these: **Rewrite the quadratic equation in standard form. Then, solve. Check your answer.**

1) $n^2 + 2n = 15$

STANDARD FORM: ________________

n = __________________

2) $9b^2 - 20 = 3b$

STANDARD FORM: ________________

b = __________________

3) $m^2 - 4m - 16 = 5$

STANDARD FORM: ________________

m = __________________

4) $2h^2 - 5h - 4 = 2h$

STANDARD FORM: ________________

h = __________________

• Writing Quadratic Equations in Standard Form • ☆

Before solving a quadratic equation, rewrite the quadratic equation by moving all terms to one side so that the quadratic equation is in **standard form**.

EXAMPLE: **Rewrite the quadratic equation in standard form. Then, solve.**

$$5x^2 + 3x - 11 = 2x^2 + x - 3 \longrightarrow 3x^2 + 2x - 8 = 0$$

$$(3x - 4)(x + 2) = 0$$

$$x = \frac{4}{3} \text{ or } x = {}^{-}2$$

Rewrite the quadratic equation in standard form. Then, solve. Check your answer.

1) $3q^2 + 6q = 2q + 15$

STANDARD FORM: ________________

q = __________________

2) $2d^2 - 3d - 3 = d^2 - d$

STANDARD FORM: ________________

d = __________________

3) $4w^2 - 3w + 3 = 2w + 2$

STANDARD FORM: ________________

w = __________________

4) $4c^2 - 3c = c - 1$

STANDARD FORM: ________________

c = __________________

• Writing Quadratic Equations in Standard Form • ☆

Rewrite the quadratic equation in standard form. Then, solve. Check your answer.

1) $x^2 + 3x = 4x + 20$

STANDARD FORM: ________________

x = ________________

2) $6k^2 + 13k = 5$

STANDARD FORM: ________________

k = ________________

3) $r^2 - 2 = r$

STANDARD FORM: ________________

r = ________________

4) $3m^2 + 7m = 2m + 2$

STANDARD FORM: ________________

m = ________________

5) $3j^2 - 5j + 1 = j^2$

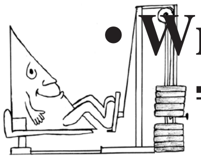
STANDARD FORM: ________________

j = ________________

6) $9q^2 - 3q + 1 = 9q - 2$

STANDARD FORM: ________________

q = ________________



• Writing Quadratic Equations in Standard Form •

Solve. Check your answer.

1) $b^2 - 4b = 5$

$b =$ ________________

2) $x^2 - 2x = 3$

$x =$ ________________

3) $5w^2 + w - 3 = 3w^2 + 5w$

$w =$ ________________

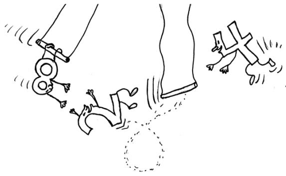
4) $3v^2 = 5v + 2$

$v =$ ________________

Challenge:

5) $p^3 + 4p^2 - 13p = 5p^2 - p$

$p =$ ________________



• The Square Root of a Number •

☆

1) The square root of a number is 7. What is the number? ________

2) The square root of a number is 5. What is the number? ________

3) The square root of a number is 8. What is the number? ________

4) The square root of a number is 3. What is the number? ________

5) The square root of a number is 12. What is the number? ________

6) If $\sqrt{\boldsymbol{n}} = 2$, then $\boldsymbol{n} =$ ________ .

7) If $\sqrt{\boldsymbol{z}} = 9$, then $z =$ ________ .

8) If $\sqrt{\boldsymbol{t}} = 4$, then $\boldsymbol{t} =$ ________ .

9) If $\sqrt{\boldsymbol{n}} = 10$, then $\boldsymbol{n} =$ ________ .

10) If $\sqrt{\boldsymbol{r}} = 6$, then $\boldsymbol{r} =$ ________ .

11) If $\sqrt{\boldsymbol{d}} = 11$, then $\boldsymbol{d} =$ ________ .

12) If $\sqrt{\boldsymbol{x}} = c$, then $\boldsymbol{x} =$ ________ , for $c \geq 0$.

• Radical Equations •

A RADICAL EQUATION contains one or more radical expressions. In this packet, the radical equations you will encounter are **square root equations**.

To solve a **square root equation**, the square root expression must be **isolated** on one side of the equation. When the square root expression is isolated, square both sides of the equation to solve.

EXAMPLE 1:

$$\sqrt{x} = 5$$
$$(\sqrt{x})^2 = 5^2$$
$$x = \mathbf{25}$$

Check the solution: $\sqrt{25} \overset{\checkmark}{=} 5$

EXAMPLE 2:

$$\sqrt{2m + 1} = 3$$
$$(\sqrt{2m + 1})^2 = 3^2$$
$$2m + 1 = 9$$
$$m = \mathbf{4}$$

Check the solution: $\sqrt{2(4) + 1} \overset{?}{=} 3$

$$\sqrt{9} \overset{\checkmark}{=} 3$$

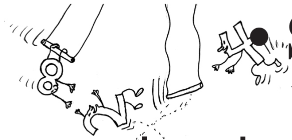
Try these: **In each exercise, solve the radical equation.**

1) $\sqrt{t} = 11$

$t =$ ____________

2) $\sqrt{4m - 3} = 5$

$m =$ ____________



• Solving Radical Equations *Practice* • ☆

In each exercise, solve the radical equation.

1) $\sqrt{h} = 8$

$h =$ ____________

2) $\sqrt{2n - 5} = 1$

$n =$ ____________

3) $\sqrt{b} = 4$

$b =$ ____________

4) $\sqrt{c - 4} = 3$

$c =$ ____________

5) $\sqrt{6z - 3} = 9$

$z =$ ____________

6) $\sqrt{k} = 5$

$k =$ ____________

7) $\sqrt{v} = 6$

$v =$ ____________

8) $\sqrt{7m + 4} = 5$

$m =$ ____________

• Isolating the Radical Expression •

Before the **square root equation** can be solved, the square root expression must be **isolated** on one side of the equation.

EXAMPLE:

$3\sqrt{2x - 5} + 4 = 10$	$3\sqrt{2x - 5} = 6$	$\sqrt{2x - 5} = 2$
The square root expression must be isolated before solving the equation.	Subtract 4 from both sides of the equation.	Divide both sides of the equation by 3 to **isolate** the square root expression.

Try these: **In each exercise, isolate the radical expression. Then, solve the radical equation.**

1) $2\sqrt{4n + 1} - 3 = 7$

$n =$ ____________

2) $5 - \sqrt{g - 8} = 4$

$g =$ ____________

3) $\frac{1}{2}\sqrt{4p} = 5$

$p =$ ____________

4) $6\sqrt{3b - 2} + 4 = 10$

$b =$ ____________

• Solving Radical Equations – Extraneous Solutions •

Sometimes, one or more solutions that you find for a radical equation will form an invalid equation when substituted into the original equation. These are **EXTRANEOUS SOLUTIONS** and they are **not** solutions to the equation.

EXAMPLE:

$\sqrt{x-4} - x = {}^-6$	$\sqrt{x-4} = x - 6$ $x - 4 = x^2 - 12x + 36$	$x^2 - 13x + 40 = 0$ $(x-5)(x-8) = 0$ $x = 5$ or $x = 8$
A square root equation.	Isolate the square root expression. Then, square both sides of the square root equation.	Write the quadratic equation in standard form. Then, solve.

Check the solutions:

$$\sqrt{5-4} - 5 \stackrel{?}{=} {}^-6 \qquad \sqrt{8-4} - 8 \stackrel{?}{=} {}^-6$$
$$\sqrt{1} - 5 \stackrel{?}{=} {}^-6 \qquad \sqrt{4} - 8 \stackrel{?}{=} {}^-6$$
$$1 - 5 \stackrel{?}{=} {}^-6 \qquad 2 - 8 \stackrel{?}{=} {}^-6$$
$${}^-4 \stackrel{\times}{=} {}^-6 \qquad {}^-6 \stackrel{\checkmark}{=} {}^-6$$

Because an invalid equation results, $x = 5$ is an **EXTRANEOUS SOLUTION**. The only valid solution is $x = 8$.

To explore why we may sometimes find **EXTRANEOUS SOLUTIONS**, examine the equation after the square root expression has been isolated: $\sqrt{x-4} = x - 6$.

The square root of a real number must be greater than or equal to zero, so the expression on the right side must be greater than or equal to zero: $x - 6 \geq 0$

Adding 6 to both sides to solve the inequality results in $x \geq 6$.

Since the found solution $x = 5$ does not satisfy $x \geq 6$, it is an **EXTRANEOUS SOLUTION**!

Try this: **Solve the radical equation. Check for extraneous solutions. If there are no solutions, write "no real solutions."**

1) $5 + \sqrt{r+1} = r$

$r =$ ______________________

• Solving Radical Equations • ☆

In each exercise, solve the radical equation. Check for extraneous solutions. If there are no real solutions, write "no real solutions."

1) $3\sqrt{8h} = 12$

$h =$ ______________________

2) $\sqrt{2k - 6} = k - 3$

$k =$ ______________________

3) $^{-}\sqrt{3n} + 10 = 1$

$n =$ ______________________

4) $\sqrt{4b^2 + 9} = 2b + 3$

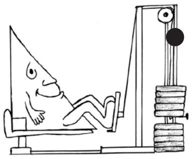
$b =$ ______________________

5) $\sqrt{5d + 6} - 2 = d$

$d =$ ______________________

6) $\sqrt{1 - 2t} = 9$

$t =$ ______________________



• Mastery Check: Radical Equations •

☆

In each exercise, solve the radical equation. Check for extraneous solutions. If there are no real solutions, write "no real solutions."

1) $\sqrt{3x + 7} = 2x$

$x =$ ____________________

2) $\sqrt{2x + 5} - 3 = x$

$x =$ ____________________

3) $2\sqrt{d - 4} = 6d$

$d =$ ____________________

4) $\sqrt{n - 1} - 7 = {}^{-}1$

$n =$ ____________________

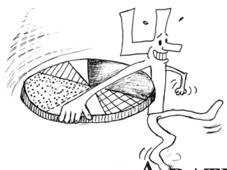
Challenge:

5) $\sqrt{z}\sqrt{z - 7} = 12$

$z =$ ____________________

6) $\sqrt[3]{2k + 7} = {}^{-}3$

$k =$ ____________________



• Rate Problems •

A **RATE** is a type of ratio that compares two quantities with different units of measure by division. A **RATE** expresses constant increase or decrease in one unit as compared to the increase or decrease in the other, like distance over time or cost over weight. **RATES** can be useful tools to solve problems like the following:

EXAMPLE: Christine scores 2 points on an exercise app for each 5 kilometers she runs. She wins a prize for every 100 points she gets. How many kilometers does Christine need to run in order to win a prize?

Christine's rate of points per kilometers is $\frac{2\text{ points}}{5\text{ kilometers}}$. So, the number of points she needs divided by the number of kilometers she must run to earn them will make an equivalent fraction.

× 50

$$\frac{2\text{ points}}{5\text{ kilometers}} = \frac{100\text{ points}}{x\text{ kilometers}}$$

× 50

Since Christine earns 2 points for every 5 kilometers and needs to earn 2 points 50 times to get to 100, she also needs to run 5 kilometers 50 times.

$$5\text{ kilometers} \bullet 50 = 250\text{ kilometers}$$

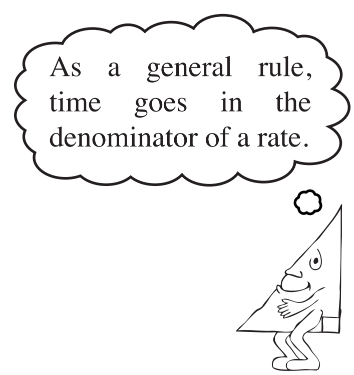
$$x = 250\text{ kilometers}$$

Christine must run 250 kilometers to earn 100 points and win a prize.

Try these:

1) Talia can build 2 chairs in 3 hours. How many can she build in 21 hours?

____



2) If a train traveling at a constant speed can go 350 miles in 2 hours, exactly how far will the train travel in 6 hours?

____

• Combining Rates •

When two rates affect the same scenario at the same time, we combine the rates by adding or subtracting them.

Example: Lena uses 25 kilowatt-hours (kWh) of electricity each week, and her sister Marijke uses 31 kWh per week. How long will it be before the sisters use 800 kWh in total?

Lena's rate of electricity use is $\frac{25 \text{ kWh}}{1 \text{ week}}$. Marijke's rate is $\frac{31 \text{ kWh}}{1 \text{ week}}$. When we add the two rates together, we have their combined rate:

$$\frac{25 \text{ kWh}}{1 \text{ week}} + \frac{31 \text{ kWh}}{1 \text{ week}} = \frac{56 \text{ kWh}}{1 \text{ week}}$$

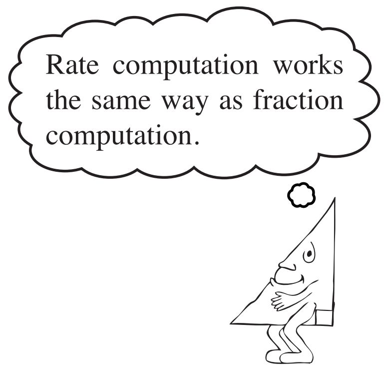
Their combined rate of electricity usage is 56 kWh per week. This rate will be equal to the rate at which the sisters use 800 kWh of electricity, so we can set the rates equal to each other and cross multiply to solve for the unknown amount of time:

$$\frac{56 \text{ kWh}}{1 \text{ week}} = \frac{800 \text{ kWh}}{\boldsymbol{x} \text{ weeks}}$$

$$56\boldsymbol{x} = 800$$

$$\boldsymbol{x} = 800 \div 56$$

$$\boldsymbol{x} = 14\tfrac{2}{7} \text{ weeks}$$



It will take the sisters 14 weeks and 2 days to use 800 kWh of electricity.

Try this:

1) Manny and Larry go to the gym together to run on treadmills. Manny runs 7 miles per hour, and Larry runs 9 miles per hour. How long will it take them to collectively run 18 miles?

____

• Combining Rates •

1) Austin can paddle a canoe 2 miles per hour. Allison can paddle a canoe 3 miles per hour. How long will it take them to canoe 28 miles if they paddle together in still water?

2) Arlo can shell 3 pea pods in a minute. Carla can shell 5 pea pods in a minute. If they shell the peas together, how many pods can they go through in an hour?

3) Matthew gets 10 new books each month. Since he has so many, he also gives away 8 books every month. He has room on his shelves for 36 more books. How long will it be before his bookshelves are full?

• Combining Rates •

Example: It takes Alice 45 minutes to complete a jigsaw puzzle. It takes Keisha an hour to complete a jigsaw puzzle of the same size. If they work together on the same jigsaw puzzle, how long will it take them to complete it?

Alice's puzzle-solving rate is $\frac{1 \text{ puzzle}}{45 \text{ minutes}}$. Keisha's rate is $\frac{1 \text{ puzzle}}{60 \text{ minutes}}$. Let's start by finding the rate at which they can solve jigsaw puzzles together.

$$\frac{1 \text{ puzzle}}{45 \text{ minutes}} + \frac{1 \text{ puzzle}}{60 \text{ minutes}}$$

Since we have two fractions with unlike denominators, our next step will be to find the LCM of the denominators, then use it to rename the fractions. The LCM of 45 and 60 is 180, so we're going to convert both fractions to 180ths.

$$\frac{4 \text{ puzzles}}{180 \text{ minutes}} + \frac{3 \text{ puzzles}}{180 \text{ minutes}} = \frac{7 \text{ puzzles}}{180 \text{ minutes}}$$

It takes the pair 180 minutes, or 3 hours, to complete 7 puzzles. But we want to know how long it'll take them to finish 1 puzzle, so we need to find the equivalent fraction.

$$\frac{7 \text{ puzzles}}{180 \text{ minutes}} = \frac{1 \text{ puzzle}}{\boldsymbol{x} \text{ minutes}}$$

Now, let's use cross products to find the value of $\boldsymbol{x}$.

$$7\boldsymbol{x} = 180$$

$$\boldsymbol{x} = \frac{180}{7}$$

$$\boldsymbol{x} = 25\tfrac{5}{7} \text{ minutes}$$

It takes Alice and Keisha $25\frac{5}{7}$ minutes to complete the puzzle when they work together.

***Try this*:**

1) Tara and Willow are sharing a bucket of popcorn. Tara usually takes 20 minutes to finish a bucket of popcorn, and Willow usually takes 30 minutes.

a) What is Tara and Willow's combined popcorn eating rate?

____

b) How long will it take them to finish the popcorn together?

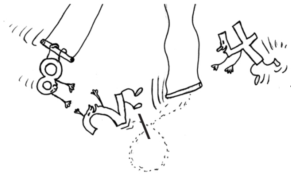
____

• Combining Rates •

1) Millie can decipher 5 encrypted messages in 6 hours. Lewis can decipher 2 encrypted messages in 3 hours. How long will it take them to decipher a message if they work together?

2) A pipe can fill a tank in 3 hours. A second pipe can fill the tank in 5 hours. How long will it take to fill the tank with both pipes running at the same time?

3) Holly can construct 4 model airplanes in 5 hours, and Jacob can construct 7 model airplanes in 10 hours. How long will it take them to build a model airplane if they work on it together?



• Combining Rates •

1) Daniel can clean his entire house in 10 hours. Rob can clean the same house in 5 hours. How long will it take them to clean the house if they work together?

____

2) Ash can dig 5 holes in 2 hours, and Bruce can dig 9 holes in 5 hours. How many holes can they dig in 15 hours if they work together?

____

3) It takes 8 minutes to fill a bathtub and 12 minutes to drain it. How long will it take to fill the bathtub if the drain is accidentally left open the whole time the water is running?

____

• Combining Rates •

1) Maggie can read 22 words in 8 seconds, and Frannie can read 24 words in 10 seconds. If they both read for a minute, how many words will they have read in total?

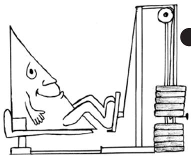
____

2) Tillie's pie-eating contest record is 4 whole pies in 30 minutes. Ethan's record is 4 pies in 20 minutes. If Tillie and Ethan share a pie, how long will it take them to eat the whole thing?

____

3) Leslie can wash 6 cars in 15 minutes. Harry can wash 5 cars in 12 minutes. How many cars can they wash if they work together for 30 minutes?

____



• **Mastery Check:** Combining Rates •

1) A bee can visit 10 flowers per minute. Another bee can visit 12 flowers per minute. How long will it take the bees to visit all 165 flowers in a garden?

____

2) A slow printer can print 100 fliers in 30 minutes. A fast printer can print 100 fliers in 20 minutes. How long will it take to print 100 fliers if both printers are going at the same time?

____

3) Jake can mow a lawn in 3 hours. Kellie can mow a lawn in 4 hours. If they work together, how long will it take them to mow one lawn?

____

Challenge:

4) Brian and Ian are decorating sugar cookies, and Alex is sneakily eating the sugar cookies. It takes Brian 2 hours to decorate two dozen cookies by himself, and it takes Ian 5 hours to decorate the same amount. Alex eats 1 cookie every hour and 15 minutes. How long will it take Brian and Ian to decorate two dozen sugar cookies?

____

• What Are the Parts? What Is the Whole? •

"The whole equals the sum of its parts."

PART + PART = WHOLE

1) 12 pounds of white sand is mixed with 4 pounds of black sand.

The first "part" in the mixture is __________ pounds of white sand.

The second "part" in the mixture is __________ pounds of black sand.

The "whole" mixture is __________ pounds of sand.

__________% of the sand mixture is white.

__________% of the sand mixture is black.

---

2) A quarter of a liter of bubble bath liquid is mixed into a bath. The water and bubble bath liquid mixed together make 100 liters.

The first "part" in the mixture is __________ liter of bubble bath liquid.

The second "part" in the mixture is __________ liters of water.

The "whole" mixture is __________ liters.

__________% of the mixture is bubble bath liquid.

__________% of the mixture is water.

---

3) 1 ounce of pigment is mixed into 15 ounces of paint medium.

The first "part" in the mixture is __________ ounce of pigment.

The second "part" in the mixture is __________ ounces of medium.

The "whole" mixture is __________ ounces of paint.

__________% of the paint is pigment.

__________% of the paint is medium.

• The Value of Each Part •

We can find the value of a part by multiplying the whole by an applicable rate or percentage.

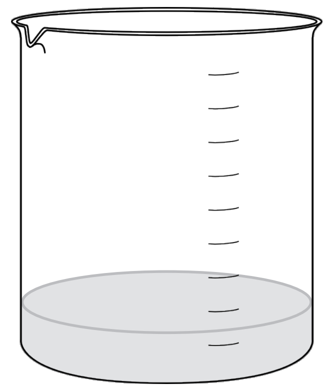
Example: The beaker pictured below holds 20 fluid ounces. If it is 15% filled with acid, how many fluid ounces of acid are in the beaker?

To answer this question, we multiply the capacity of the beaker by the percentage to find the part of the beaker that is full.

"Whole • Rate = Part"

20 fluid ounces • 15% = 3 fluid ounces

There are 3 fluid ounces of acid in the beaker.



Try these: **Find the quantity.**

1) How much pure juice is there in 2 gallons of 12.5% juice drink?

____

2) How much pure saline is in 4 liters of a 16% saline solution?

____

3) How much fat is there in 200 gallons of 2% milk?

____

4) How many ounces of pigment are there in a gallon of paint that is 12.5% pigment? (A gallon is 128 fluid ounces.)

____

• Wholes & Parts of Mixtures •

☆

A mixture of nuts contains 5 pounds of peanuts that cost $4 per pound and 3 pounds of cashews that cost $8 per pound, resulting in a mixture of nuts that is worth $5.50 per pound.

1) What are the "parts" in this scenario? What is the "whole"?

Part: ______________________________

Part: ______________________________

Whole: ______________________________

2) What is the total cost of the peanuts?

____

3) What is the total cost of the cashews?

____

4) What is the total weight of the entire mixture? What is its total cost?

There are ______________ pounds of nuts costing $ ______________.

5) Write an equation that represents the scenario:

____ LB • $____ PER LB + ____ LB • $____ PER LB = ____ LB • $____ PER LB

PEANUTS **CASHEWS** **MIXTURE**

6) **TRUE** or **FALSE**: The total weight on one side of the equation is equal to the total weight on the other side of the equation.

7) **TRUE** or **FALSE**: The total cost on one side of the equation is equal to the total cost on the other side of the equation.

• Wholes & Parts of Mixtures •

EXAMPLE: Raisins cost $4.00 per pound and almonds cost $7.00 per pound. Twenty pounds of raisins are mixed with 10 pounds of almonds. What is the cost per pound of the resulting mixture?

First, we need to identify the wholes and the parts. One part is 20 pounds of raisins, the other part is 10 pounds of almonds, and the whole is the sum of the raisins and the almonds.

The value of each part is its amount multiplied by its cost per pound.

20 pounds • $4.00 per pound + 10 pounds • $7.00 per pound

The same goes for the whole. The total amount of mixture is 30 pounds, and the cost per pound is unknown.

30 pounds • ***x*** per pound

Now, since we know that the whole is equal to the sum of its parts, we can write a full equation to solve.

20 LB • $4.00 PER LB + 10 LB • $7.00 PER LB = 30 LB • ***x*** PER LB

$80 + 70 = 30x$

$x = \$5.00$

The cost per pound of the resulting mixture is $5.00.

Try this:

1) Soda water costs 50¢ per quart. Juice concentrate costs $1.20 per quart. What is the cost per quart of a fizzy juice drink made of 4 quarts of soda water and 1 quart of juice concentrate?

_____ QT • $_____ PER QT + _____ QT • $_____ PER QT = _____ QT • $_____ PER QT

____

• Wholes & Parts of Mixtures •

The equation below represents what happens when we mix 2 liters of 10% juice with 6 liters of 30% juice.

2 LITERS • 10% JUICE + 6 LITERS • 30% JUICE = ____ LITERS • ____% JUICE

1) What are the “parts” in this equation? What is the “whole”?

Part: ______________________________

Part: ______________________________

Whole: ______________________________

2) What is the amount of pure juice in 2 liters of 10% juice?

____

3) What is the amount of pure juice in 6 liters of 30% juice?

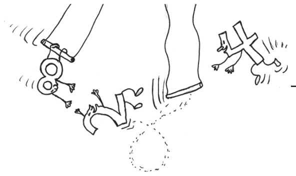
____

4) What is the total amount of pure juice in the whole mixture?

____

5) What percentage of the resulting mixture is pure juice?

____



• Wholes & Parts of Mixtures •

A mixture contains 6 pounds of peanuts that cost $3 per pound and 3 pounds of cashews that cost $9 per pound.

1) What are the "parts" in this equation? What is the "whole"?

____

____

2) Write an equation that represents the wholes and parts of the mixture.

_____ LB • $_____ PER LB + _____ LB • $_____ PER LB = _____ LB • $_____ PER LB

3) What is the cost per pound of the resulting mixture?

____

A mixture contains 4 liters of 20% cranberry juice and 4 liters of 50% cranberry juice.

4) What are the "parts" in this equation? What is the "whole"?

____

____

5) Write an equation that represents the wholes and parts of the mixture.

____ LITERS • ____% JUICE + ____ LITERS • ____% JUICE = ____ LITERS • ____% JUICE

6) What is the concentration of cranberry juice in the resulting mixture?

____

• Wholes & Parts of Mixtures •

EXAMPLE: How much of a 50% salt solution should be mixed with 10 liters of a 10% salt solution in order to create a mixture that is 25% salt?

In this scenario, the amount of one part of the mixture is unknown. The total amount must therefore be the sum of the known amount and the unknown amount (*i.e.*, 10 + x).

$$10 \text{ liters} \bullet 10\% \text{ salt} + x \text{ liters} \bullet 50\% \text{ salt} = (10 + x) \text{ liters} \bullet 25\% \text{ salt}$$

$$10 \bullet 0.1 + x \bullet 0.5 = (10 + x) \bullet 0.25$$

$$1 + 0.5x = 2.5 + 0.25x$$

$$0.25x = 1.5$$

$$x = 6 \text{ liters}$$

Six liters of 50% salt solution must be added to 10 liters of 10% salt solution to create a mixture that is 25% salt.

Try this:

1) How many liters of 30% acid must be added to 2 liters of 15% acid to result in a mixture that is 25% acid.

_____ L • _____% ACID + _____ L • _____% ACID = (_____+_____) L • _____% ACID

____

• Mixture Problem Practice •

☆

1) How many gallons of 50% grape juice must be added to 16 gallons of 25% grape juice to result in a mixture that is 40% juice?

____

2) How many ounces of 45% pigmented paint must be added to 12 ounces of 25% pigmented paint to result in a paint mixture that is $37\frac{1}{2}$% pigment?

____

3) Margaret mixes the remaining 6 cups of a 25% concentrated detergent with the remaining 4 cups of an $87\frac{1}{2}$% concentrated detergent. What is the concentration of the leftover detergent mixture?

____

4) How much 100% active ingredient powder would you need to add to 4 mL of multivitamin powder that is 20% active ingredient powder in order to strengthen the mixture to 50% active ingredient powder?

____

• Two Unknowns, One Variable •

When there are two unknown quantities in a scenario, we can either write a system in two variables or represent both unknowns using the same variable within two different expressions.

Example: An amount of 50% juice is mixed with an amount of 80% juice to make 20 cups of 75% juice. Write two expressions to represent the unknown volume of each kind of juice.

When we eventually write an equation to solve for the unknown volumes of each ingredient, we will use x in place of the unknown volume of 50% juice.

Amount of 50% juice: x

Now, let's use the same variable to write an expression that represents the other unknown quantity. In any whole-and-parts scenario, each part is equal to the whole minus the other parts. The whole in this scenario is 20 cups, and the other part is x.

Amount of 80% juice: $20 - x$

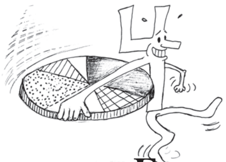
Try this: **Write two expressions to represent the two unknowns in each scenario below. Use only one variable.**

1) An amount of 5% acid is mixed with an amount of 40% acid to make 10 liters of a 20% acid solution.

Amount of 5% acid: __________

Amount of 40% acid: __________

Amount of 20% acid: __________



• Mixture Problems •

Example: How much of a 50% juice and an 80% juice should be mixed together to make 20 liters of a 75% juice?

In this scenario, both amounts of the parts are unknown, but the amount of the whole mixture is known. So, we know that the sum of the parts must be 20 liters.

First, let's solve for the amount of the 50% juice that goes into the mixture. We'll call that amount $\boldsymbol{x}$. Since the whole is the sum of the parts, that means that the other amount must be $\boldsymbol{x}$ less than 20 liters.

$$\boldsymbol{x} \text{ liters} \bullet 50\% \text{ juice} + (20 - \boldsymbol{x}) \text{ liters} \bullet 80\% \text{ juice}$$

Now we can write an equation to solve for the unknown amount of 50% juice.

$$\boldsymbol{x} \bullet 0.5 + (20 - \boldsymbol{x}) \bullet 0.8 = 20 \bullet 0.75$$

$$0.5\boldsymbol{x} + 16 - 0.8\boldsymbol{x} = 15$$

$$^{-}0.3\boldsymbol{x} = {}^{-}1$$

$$\boldsymbol{x} = 3\tfrac{1}{3} \text{ liters}$$

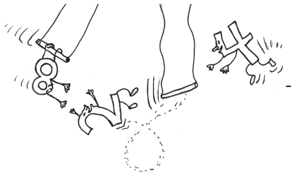
So, if $3\frac{1}{3}$ liters of 50% juice go into the mixture, then $16\frac{2}{3}$ liters of 80% juice must go into the mixture to make a total of 20 liters of 75% juice.

Try this:

1) How much 30% acetone and 80% acetone should be mixed together to have 50 cups of 60% acetone?

____

____



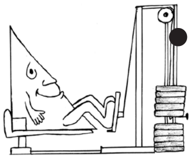
• Mixture Problem Practice •

1) 24-karat gold is made of 100% pure gold. How much 12-karat gold and how much 6-karat gold should be smelted together to create 20 grams of metal that is exactly 40% gold?

2) How many cups of 25% dye must be added to 16 cups of $6\frac{1}{4}\%$ dye to result in a mixture that is $12\frac{1}{2}\%$ dye?

3) How much 75% concentrated cleaning solution and 100% concentrated cleaning solution should be mixed together to have 16 cups of 80% concentrated cleaning solution?

4) A glass of chocolate milk was made from a mixture of 2 fluid ounces of 5% chocolate syrup and 8 fluid ounces of 20% chocolate syrup. What is the percentage of milk in the glass of chocolate milk?



• Mastery Check: Mixture Problems •

1) What is the concentration of acid in a mixture of 10 cups of 50% acid and 14 cups of 20% acid?

____

2) How many milliliters of paste that is 25% recycled plastic must be added to 40 milliliters of paste that is 10% recycled plastic to result in a mixture that is 20% recycled plastic?

____

3) There is a 10% salt solution and a 30% salt solution. How much of each is needed to make 10 liters of a mixture that is 25% salt?

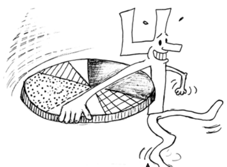
____

____

Challenge:

4) 5 liters of 10% salt solution are mixed with 15 liters of 30% salt. How much water must evaporate from the mixture so that it is 40% salt solution?

____



• Two–Way Frequency Tables •

Direct Teaching

A **TABLE** is a chart that is used to organize data. The word "frequency" means "the number of times something happens." A **FREQUENCY TABLE** shows how often each value comes up in a list of data. A **TWO–WAY FREQUENCY TABLE** shows two variables and their frequencies. The frequencies of one variable are shown in the table's rows and the frequencies of the other variable are shown in its columns.

EXAMPLE: **The Paws Not Claws Veterinary Clinic takes in dogs and cats. The table below shows the animals and their colors currently housed at the clinic.**

		Color of the Animal						
		Black	**White**	**Brown**	**Gray**	**Orange**	**Multi–Colored**	**Total**
Animal	**Dogs**	2	1	7	1	0	5	16
	Cats	5	2	2	1	3	4	17
	Total	7	3	9	2	3	9	

This **TWO–WAY FREQUENCY** table shows the types of animal (dog and cat) in its rows and the colors of the animals (black, white, brown, gray, orange, and multicolored) in its columns. For example, there are 3 orange cats and 7 brown dogs.

It also has a column for the total number of each color and a row for the total number of each animal. Under the "Black" column there are 2 black dogs and 5 black cats. 2 + 5 = 7, so there are 7 black animals altogether. Across the "Dogs" row there are 2 black dogs, 1 white dog, 7 brown dogs, 1 gray dog, 0 orange dogs, and 5 multicolored dogs. 2 + 1 + 7 + 1 + 0 + 5 = 16, so there are 16 dogs altogether.

The cell where the "Total" row and the "Total" column intersect is where the total number of animals at the veterinary clinic will be written.

***Try this*:** **Solve the following problem using the table above.**

1) Fill in the total number of animals at the veterinary clinic in the appropriate cell.

• Two–Way Frequency Tables •

Melissa wants to grow her pizzeria, We Knead the Dough, so she is analyzing her sales within the past month to determine how she wants to spend her marketing budget for this month. She only sells pizza by the slice.

		Type of Pizza				
		Cheese	Pepperoni	The Works	Veggie	Total
Pizza Style	California	147	247	146	359	
	New York	532	511	241	200	
	Chicago	245	230	86	98	
	Total					

1) Fill in the numbers of each style of pizza slice sold, the numbers of each type of pizza slice sold, and the total number of pizza slices sold last month at We Knead the Dough.

2) How many slices of cheese Chicago–style pizza were sold? ___________

3) How many slices of pepperoni California–style pizza were sold? ___________

4) How many slices of cheese pizza were sold? ___________

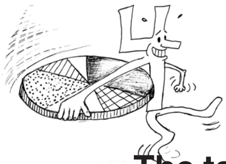
5) How many slices of veggie pizza were sold? ___________

6) How many slices of New York–style pizza were sold? ___________

7) What is the most popular pizza slice? ___________________________________

8) What is the least popular pizza slice? ___________________________________

9) How many slices of pizza were sold altogether last month? ___________



• Two–Way Frequency Tables •

☆

The table below shows the results of a random sample of a group of people's hair and eye colors that Melanie took for her genealogy class.

1) Fill in the empty cells with the missing numbers.

		Hair Color				
		Blonde	**Brown**	**Black**	**Red**	**Total**
Eye Color	**Blue**	8	14			24
	Brown	3	22	14		39
	Green		0	0	2	
	Hazel	3		1	0	13
	Total	15		16	3	

The table below shows the pastry and drink combinations served for breakfast this morning at Le Café d'Alice.

2) Fill in the empty cells with the missing numbers.

		Food				
		Croissant	**Crêpe**	**Brioche**	**Éclair**	**Total**
Drink	**Tea**	0		5	0	
	Coffee	4	7	5		25
	Hot Chocolate		4		0	7
	Total	6	19			

• Two–Way Frequency Tables • ☆

Direct Teaching

Let's look at more complex questions about TWO–WAY FREQUENCY TABLES.

EXAMPLE: **Jane goes to chess club each day after school. Sometimes she plays with white pieces and sometimes she plays with black pieces. The table below shows the number of games she played each day and how many games of each color she used. How many games did she play on Monday or Friday with black pieces?**

		Day of the School Week					
		Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**	**Total**
Color	**White**	1	3	4	2	0	10
	Black	**3**	0	1	2	**6**	12
	Total	4	3	5	4	6	22

She played 3 games on Monday with black pieces and 6 games on Friday with black pieces. We can add these numbers together to determine how many games on Monday or Friday she played with black pieces.

3 + 6 = 9, so Jane played 9 games on Monday or Friday with black pieces.

Try these: **Solve the following problems using the table above.**

1) How many games did Jane play on Wednesday or Thursday with white pieces?

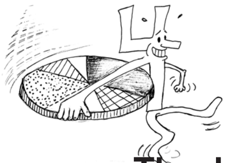
____

2) How many games did Jane play on Monday or Tuesday with white pieces?

____

3) How many games did Jane play on Tuesday or Thursday with black pieces?

____



• Two–Way Frequency Tables •

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The biology department at Martinek University has its students dissect organs of various species throughout the school year.

		Organ				
		Brain	Heart	Lung	Eyeball	Total
Species	Frog	7	9	4	0	20
	Cow	15	8	0	13	36
	Fetal Pig	6	4	2	0	12
	Rabbit	1	5	1	0	7
	Total	29	26	7	13	75

1) How many fetal pig organs were dissected? __________

2) How many eyeballs were dissected? __________

3) How many rabbit or cow brains were dissected? __________

4) How many frog or rabbit lungs were dissected? __________

5) How many fetal pig eyeballs or frog brains were dissected? __________

6) How many cow or rabbit eyeballs were dissected? __________

7) How many frog, cow, or fetal pig hearts were dissected? __________

8) How many rabbit or cow organs were dissected? __________

• Fractions and Percents •

We can find a fraction or a percentage of a certain quality, or certain qualities, out of the total number of items from a TWO–WAY FREQUENCY TABLE.

Direct Teaching

EXAMPLE: **It's a typical Wednesday night and the local jailhouse recorded a list of federal misdemeanors it processed tonight according to the type of offense and the gender of the defendant who is accused of committing it. What percentage of defendants who were processed tonight are women?**

		Type of Federal Misdemeanor			
		Impersonating a 4-H Club Member	**Depositing a Plant to be Mailed**	**Hunting on a Wildlife Refuge**	**Total**
Gender	**Women**	6	1	2	9
	Men	4	0	7	11
	Total	10	1	9	20

Since we were asked to find the percentage of defendants who are women, we need to look at the row that contains the total number of women and the cell that contains the total number of defendants overall. The table gives us the total number of women, 9, and the total number of defendants, 20.

So, the fraction of defendants who are women is $\frac{9}{20}$. As a percentage, 45% of defendants who were processed tonight are women.

Try these: **Solve the following problems using the table above.**

1) What percentage of defendants who were processed tonight are men?

____

2) What fraction of defendants who were processed tonight were impersonating a 4-H Club member or hunting on a wildlife refuge?

____

• Two–Way Frequency Tables •

☆

Chuckles the Clown wears different outfits for his performances. The table below shows how many times he wore each onesie and shoe combination during his performances last week.

		Onesie Color					
		Blue	**Pink**	**Yellow**	**Red**	**Orange**	**Total**
Shoe Color	**Green**	1	0	0	0	2	
	Pink	0	0	1	0	1	
	Orange	0	1	1	0	0	
	Purple	0	0	0	3	0	
	Total						

1) Fill in the missing totals in the table above.

2) What percentage of outfits included orange shoes and a pink onesie?

____

3) What fraction of outfits included purple shoes and a red onesie?

____

4) What fraction of outfits included pink shoes? ______________

5) What fraction of outfits included purple or green shoes? ______________

6) What percentage of outfits included a blue or pink onesie? ______________

7) What fraction of outfits included a yellow or red onesie? ______________

8) What percentage of outfits included orange, pink, or green shoes? __________

• The Complement •

Direct Teaching

The **COMPLEMENT** of a group is everything that is *not* in the group. In terms of fractions and percentages, the sum of values in a table is equal to 1 or 100%. We can find the **COMPLEMENT** of a specific group by first finding its fraction or percentage (the part), and then subtracting it from 1 or 100% (the whole).

EXAMPLE: **Every student at a high school is studying one foreign language. The table below shows the number of students in each year who are enrolled in each foreign language class. What fraction of students are not freshman or sophomores taking French?**

		Foreign Language Class				
		French	**Spanish**	**Japanese**	**German**	**Total**
Year	**Freshmen**	178	254	107	98	637
	Sophomores	222	203	98	100	623
	Juniors	168	224	112	102	606
	Seniors	130	197	109	98	534
	Total	698	878	426	398	2,400

The number of freshman and sophomores taking French is 178 + 222 = 400. The total number of students at the high school is 2,400.

So, the fraction of freshman and sophomores taking French is $\frac{400}{2,400} = \frac{1}{6}$ of the total number of students.

To find the fraction of students who are **not** freshman or sophomores taking French, we subtract this fraction from 1.

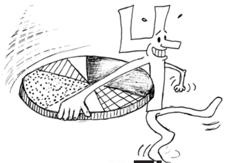
$$1 - \frac{1}{6} = \frac{5}{6}$$

Whole MINUS Part

***Try this*: Solve the following problem using the table above.**

1) What fraction of students are not juniors or seniors taking German?

____



• Two–Way Frequency Tables • ☆

The extended Bianchette family took a survey of their favorite movie genres.

		Movie Genre				
		Action	**Drama**	**Horror**	**Comedy**	**Total**
Age Range	**Adults**	1	5	2	6	
	Children	4	0	3	4	
	Total					

1) Fill in the missing totals in the table above.

2) What percentage were children who said comedy is their favorite genre?

____

3) What fraction were adults who said drama is their favorite genre? __________

4) What fraction were children who said action or horror is their favorite genre?

____

5) What percentage of the children said drama is not their favorite genre?

____

6) What fraction of the adults said horror or comedy is their favorite genre?

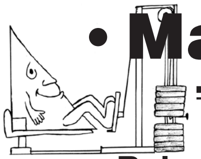
____

7) What fraction were not children who said action or drama is their favorite genre?

____

8) What percentage were not adults who said action, drama, or horror is their favorite genre?

____



• Mastery Check: Two–Way Frequency Tables • ☆

Below is a table showing the number of girls and the number of boys whose favorite class is art, music, math, science, or history.

		Favorite School Subject					
		Art	Music	Math	Science	History	Total
Gender	Girls	12	11	18	10	5	56
	Boys	8	4	6	16	10	44
	Total	20	15	24	26	15	100

1) What percentage of students are girls? _____________

2) What percentage of students say music is their favorite class? _____________

3) What fraction of boys say science is not their favorite class? _____________

4) What fraction of girls say math is their favorite class? _____________

Challenge: **Mia asked 50 people in the cafeteria line what sandwich they ordered for lunch. The values in the table are in percentages out of the total number of students surveyed.**

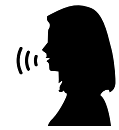
		Type of Sandwich					
		Turkey	BLT	Ham	PB & J	Egg	Total
Bread	White	10%	8%	32%	18%	4%	
	Wheat	2%	0%	14%	6%	6%	
	Total						

5) Fill in the missing total percentages in the table above.

6) How many students had a ham sandwich on wheat bread? _____________

7) How many students had an egg sandwich? _____________

• Quartiles •



Direct Teaching

QUARTILES mark *quarters* of a data set. We find the **QUARTILES** by taking the **MEDIAN** of both the lower half and upper half of the data set. The **MEDIAN** of the lower half is called the **FIRST QUARTILE** ($\mathbf{Q_1}$), and the **MEDIAN** of the upper half is called the **THIRD QUARTILE** ($\mathbf{Q_3}$). These values can help us measure the spread of the data set.

EXAMPLE: Find the **FIRST QUARTILE**, **MEDIAN**, and **THIRD QUARTILE** of the following data set:

{**5**, **1**, **2**, **7**, **3**, **8**, **5**, **2**, **3**}

First, let's put the data set in order from *least* to *greatest* and find the **MEDIAN**.

{**1**, **2**, **2**, **3**, **3**, **5**, **5**, **7**, **8**}

So, the **MEDIAN** of the data set is **3**. Now, let's look at the upper and lower halves to find the **QUARTILES**.

{**1**, **2**, **2**, **3**, **3**, **5**, **5**, **7**, **8**}

Lower Half	Upper Half
{**1**, **2**, **2**, **3**} **MEDIAN** = **2**	{**5**, **5**, **7**, **8**} **MEDIAN** = **6**

NOTE: When finding the upper and lower halves of a data set with an odd number of values, we do not include the median of the entire set.

So, the **FIRST QUARTILE** is **2**, the **MEDIAN** is **3**, and the **THIRD QUARTILE** is **6**.

Try this: **Find the FIRST QUARTILE, MEDIAN, and THIRD QUARTILE of the following data set.**

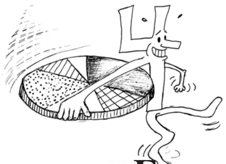
1) {11, 3, 6, 1, 2, 0, 3, 4, 7}

$\mathbf{Q_1}$ = ________

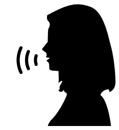
Lower Half	Upper Half

MEDIAN = ________

$\mathbf{Q_3}$ = ________



• Quartiles •



Direct Teaching

EXAMPLE: Find the **FIRST QUARTILE**, **MEDIAN**, and **THIRD QUARTILE** of the following data set:

{**23**, **20**, **33**, **27**, **39**, **36**}

First, let's put the data set in order from *least* to *greatest* and find the **MEDIAN**.

{**20**, **23**, **27**, **33**, **36**, **39**}

So, the **MEDIAN** of the data set is **30**. Now, let's look at the upper and lower halves to find the **QUARTILES**.

{**20**, **23**, **27**, **33**, **36**, **39**}

Lower Half	Upper Half
{**20**, **23**, **27**} **MEDIAN** = **23**	{**33**, **36**, **39**} **MEDIAN** = **36**

NOTE: When finding the upper and lower halves of a data set with an odd number of values, we do not include the median of the entire set. For a data set with an even number of values, we split it exactly in half.

So, the **FIRST QUARTILE** is **23**, the **MEDIAN** is **30**, and the **THIRD QUARTILE** is **36**.

Try this: **Find the FIRST QUARTILE, MEDIAN, and THIRD QUARTILE of the following data set.**

1) {31, 30, 14, 22, 23, 29}

Q_1 = ________

Lower Half	Upper Half

MEDIAN = ________

Q_3 = ________

• Finding the Interquartile Range •

Direct Teaching

The RANGE of a data set is the difference between the smallest and largest values, and its INTERQUARTILE RANGE is the difference between the THIRD QUARTILE and FIRST QUARTILE ($Q_3 - Q_1$).

EXAMPLE: Find the RANGE and INTERQUARTILE RANGE of the following data set:

{**25**, **31**, **21**, **28**, **42**, **45**, **27**, **40**, **20**}

After determining that the smallest value is 20, the largest value is 45, $Q_1 = 23$, and $Q_3 = 41$, we can find the differences.

Since 45 – 20 = 25, the RANGE of the data set is **25**, and since 41 – 23 = 18, the INTERQUARTILE RANGE of the data set is **18**.

Try these: **Find the MEDIAN, QUARTILES, RANGE, and INTERQUARTILE RANGE of the following data sets.**

1) {2, 6, 7, 11, 27, 2, 24, 18, 15}

ORDERED SET: ____

MEDIAN = __________ Q_1 = __________ Q_3 = __________

RANGE = __________

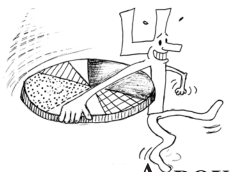
INTERQUARTILE RANGE = __________

2) {$^{-}1$, $^{-}4$, 3, 4, $^{-}8$, 0}

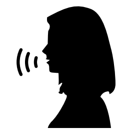
MEDIAN = __________ Q_1 = __________ Q_3 = __________

RANGE = __________

INTERQUARTILE RANGE = __________



• Box and Whisker Plots •

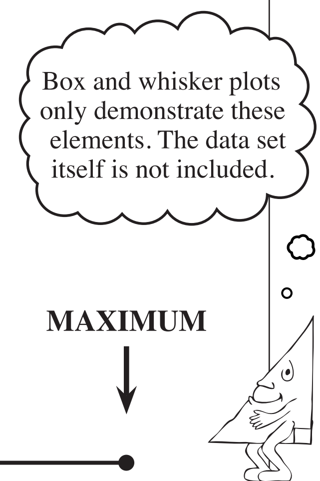


Direct Teaching

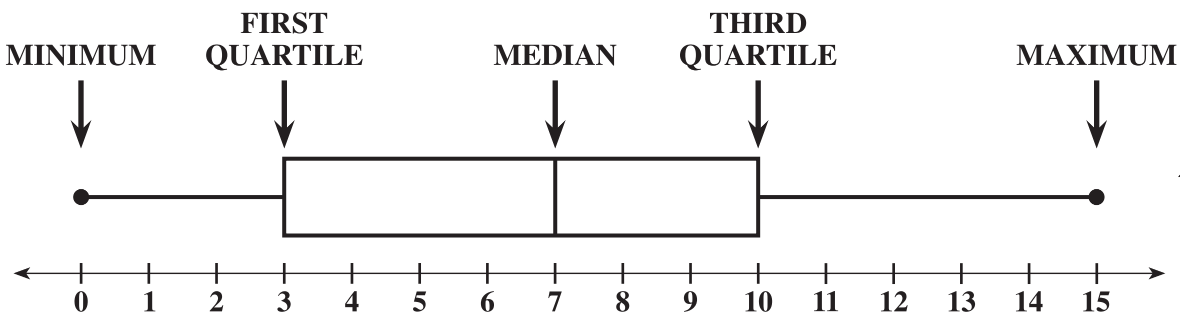
A **BOX AND WHISKER PLOT** organizes a data set into *quarters*. This is done using **MEDIAN** and **QUARTILE** values. The "box" stands for data values between the **FIRST** and **THIRD QUARTILES**, with a line at the **MEDIAN**.

Values *less than* the **FIRST QUARTILE** and *greater than* the **THIRD QUARTILE** are shown with line segments at opposite sides of the box (the "whiskers"), with points showing the **MINIMUM** and **MAXIMUM** values of the data set.

A number line is placed underneath these values for comparison.



EXAMPLE:



Looking at the **BOX AND WHISKER PLOT** above, see that the **MEDIAN** line is at **7**, the **FIRST QUARTILE** is **3**, and the **THIRD QUARTILE** is **10**.

Try these:

1) Using the data set represented by the box and whisker plot above,

a) find the minimum value. ___________

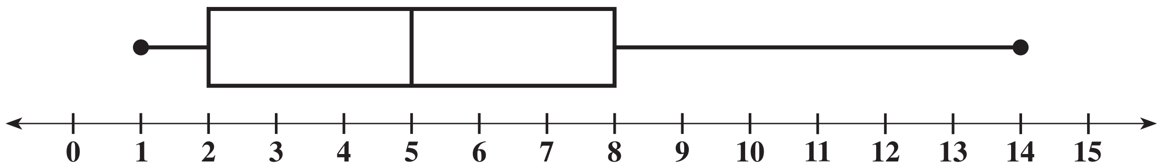
b) find the maximum value. ___________

2) What is the range? ____________

3) What is the interquartile range? ____________

4) What values are we certain are in the data set? ________________

• Box and Whisker Plots •



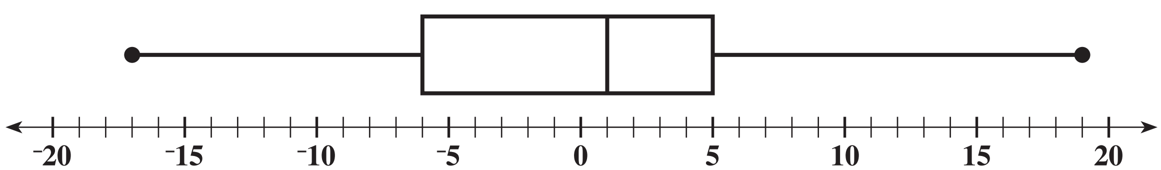
1) What is the median of the data set represented by the box and whisker plot above?

____

2) What is the first quartile? ____________ Third quartile? ____________

3) What is the range? ____________

4) What is the interquartile range? ____________



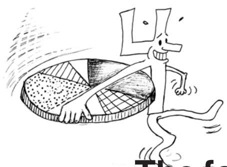
5) What is the median of the data set represented by the box and whisker plot above?

____

6) What is the first quartile? ____________ Third quartile? ____________

7) What is the range? ____________

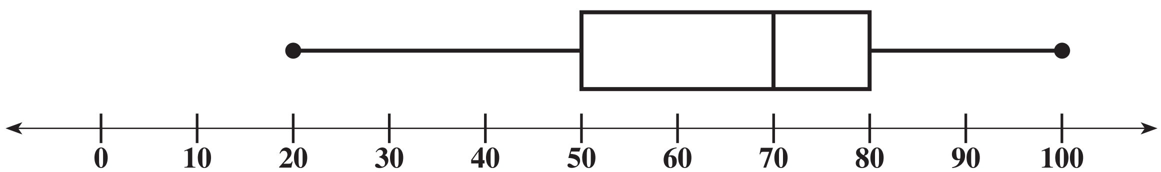
8) What is the interquartile range? ____________



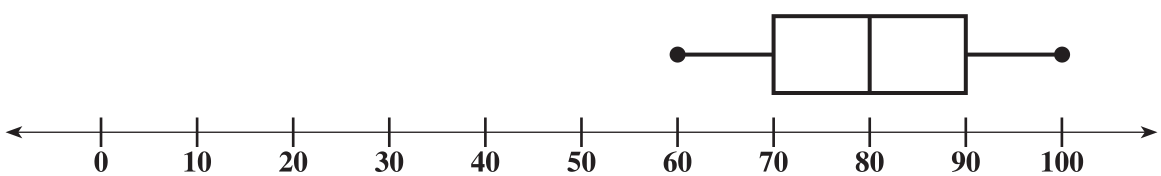
• Comparing Box and Whisker Plots •

The following box and whisker plots represent the test scores Susan received during first semester and second semester last year.

First Semester



Second Semester



1) Which data set has the larger range? _____________

2) Which data set has the larger median? _____________

By how many points? _____________

3) Which data set has the smaller interquartile range? _____________

4) Which data set has the larger third quartile? _____________

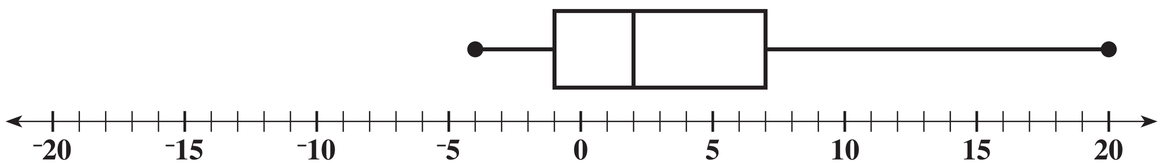
5) Which data set has the smaller first quartile? _____________

• Outliers •

A value in a data set that is much higher or much lower than the rest of the data is called an **OUTLIER**. There is a way to calculate possible **OUTLIERS** using the **INTERQUARTILE RANGE** and **QUARTILES** of a data set. If a value is at least 1.5 times the **INTERQUARTILE RANGE** outside of either **QUARTILE**, then it is considered an **OUTLIER** of the data set.

Direct Teaching

EXAMPLE: Determine if the value 20 is an outlier in the data set represented below.

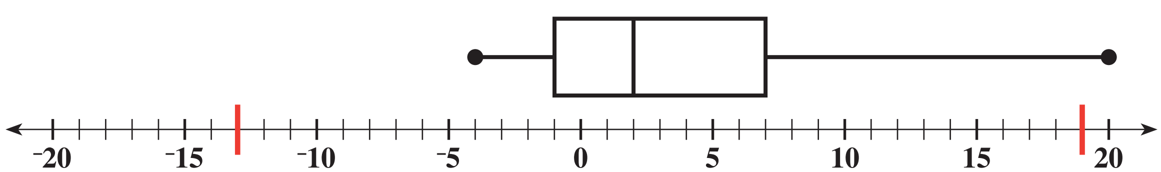


The **INTERQUARTILE RANGE** is $7 - {}^{-}1 = 8$. One and one half times this value is $1.5 \times 8 = 12$. Now we *subtract* 12 from the **FIRST QUARTILE** and *add* 12 to the **THIRD QUARTILE**.

$$Q_1 - 12 = {}^{-}1 - 12 = {}^{-}13$$

$$Q_3 + 12 = 7 + 12 = 19$$

Let's mark these values on the **BOX AND WHISKER PLOT**.

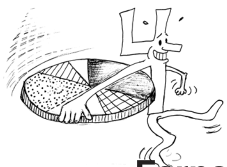


Any value in the data set represented by the **BOX AND WHISKER PLOT** that lies outside of these marked values (less than ⁻13 or greater than 19) is considered an **OUTLIER**. So, 20 is an **OUTLIER** of the data set.

Try these: **Use the box and whisker plot above to answer the following questions.**

1) Is 15 an outlier? ________ Explain. ________________________________

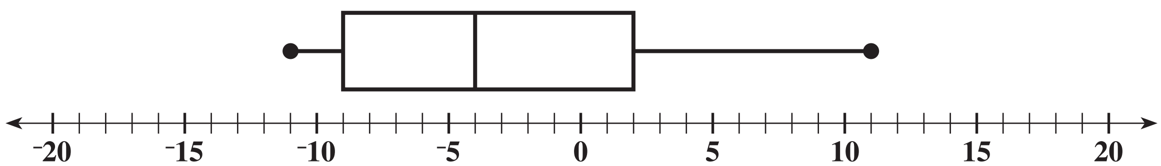
2) Is ⁻16 an outlier? ________ Explain. ________________________________



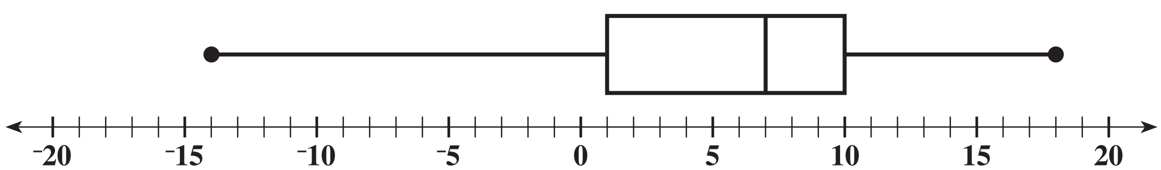
• Comparing Box and Whisker Plots •

Fernando checks his savings account at the end of every month and records how much money he gained or lost. The box and whisker plots below show the data he recorded the first half of the year and the second half of the year.

January through June



July through December



1) Which data set has the larger interquartile range? ______________________

2) Which data set has the larger median? ______________________

By how many dollars? _____________

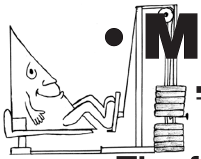
3) Which data set has the smaller range? ______________________

4) If Fernando gained $11 in February, is it an outlier? Explain.

____

5) If Fernando lost $14 in October, is it an outlier? Explain.

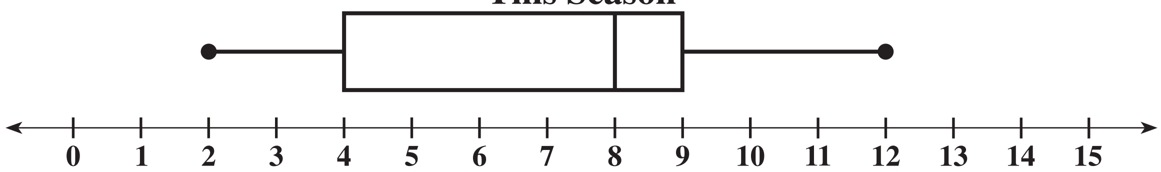
____



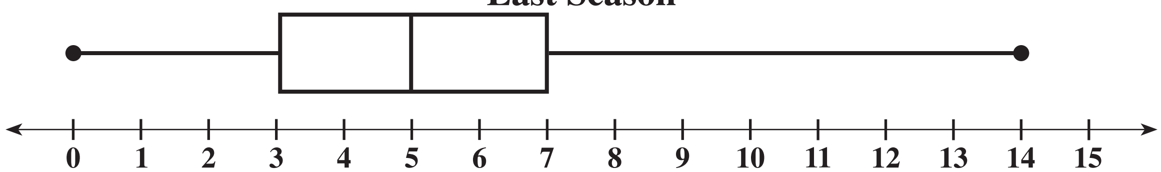
• Mastery Check: Box and Whisker Plots • ☆

The following box and whisker plots represent the number of points Rachel scored in each basketball game this season and last season.

This Season



Last Season



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1) Which season has the smaller range? ____________________

2) Which season has the larger interquartile range? ____________________

3) Which season has the smaller median? ____________________

By how many points? ____________

4) If Rachel scored 14 points during one of the games last season, is it an outlier? Explain.

____

Challenge:

5) If you were to randomly select a game that Rachel played this season, what is the probability that she scored less than 9 points during that game? Explain.

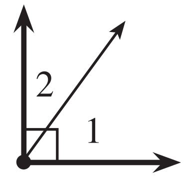
____

• Complementary, Supplementary, and Vertical Angles •

Two angles are **COMPLEMENTARY** when the sum of their measures equals 90˚.

$\angle 1$ & $\angle 2$ are **COMPLEMENTARY**.

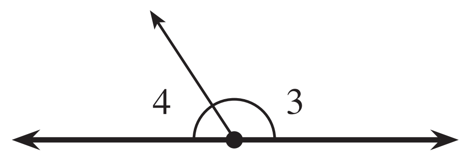
$m\angle 1 + m\angle 2 = \mathbf{90^\circ}$



Two angles are **SUPPLEMENTARY** when the sum of their measures equals 180˚. A **LINEAR PAIR** of angles is a pair of adjacent, supplementary angles.

$\angle 3$ & $\angle 4$ form a linear pair and are **SUPPLEMENTARY**.

$m\angle 3 + m\angle 4 = \mathbf{180^\circ}$

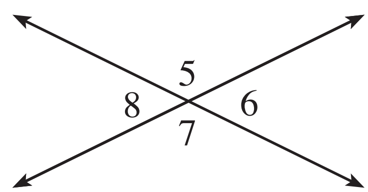


When two lines intersect at a point, four angles are formed.

The angles that are *opposite* (or *across from*) each other are known as **VERTICAL ANGLES**.

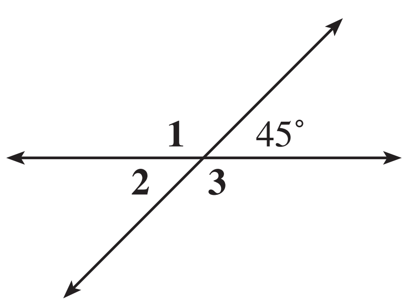
$\angle 5$ & $\angle 7$ are **VERTICAL ANGLES**, as are $\angle 6$ & $\angle 8$.

VERTICAL ANGLES are congruent.



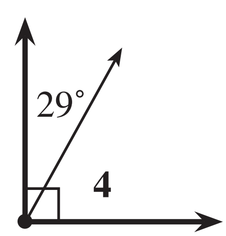
Find the missing values.

1)

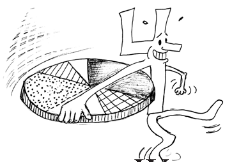


$m\angle 1$ = ________ $m\angle 2$ = ________

2)



$m\angle 4$ = ________

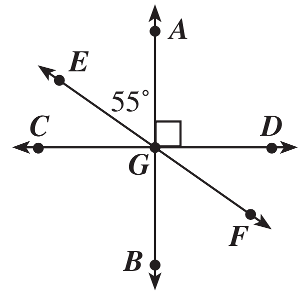


• Finding Missing Angles •

Direct Teaching

We can use what we know about complementary, supplementary, and vertical angles to figure out the measures of some missing angles.

Example:



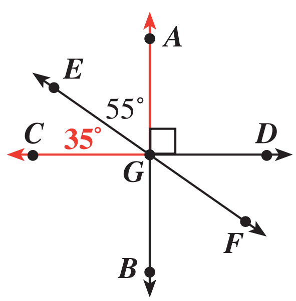
$\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$ intersect at point G. Find $m\angle AGF$.

The whole ($m\angle AGF$) is equal to the sum of its parts ($m\angle AGD$ and $m\angle DGF$). So, in order to find $m\angle AGF$, we must first find $m\angle AGD$ and $m\angle DGF$.

Since $\angle AGD$ and $\angle AGC$ form a linear pair, they are supplementary. We are given $m\angle AGD = 90^\circ$, so $m\angle AGC = 180^\circ - 90^\circ = 90^\circ$.

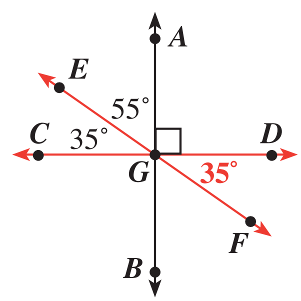
Since $\angle CGE$ and $\angle AGE$ are complementary:

$$m\angle CGE = 90^\circ - 55^\circ = 35^\circ.$$



Since vertical angles are congruent:

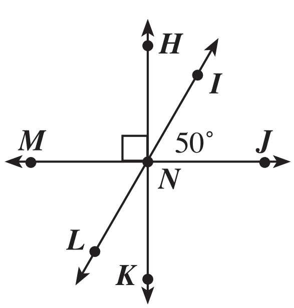
$$m\angle DGF = m\angle CGE = 35^\circ.$$



So, $m\angle AGF = m\angle AGD + m\angle DGF = 90^\circ + 35^\circ = 125^\circ$.

Try this:

1) $\overleftrightarrow{HK}$, $\overleftrightarrow{IL}$, and $\overleftrightarrow{JM}$ intersect at point N.



$m\angle HNI =$ ______, $m\angle KNL =$ ______

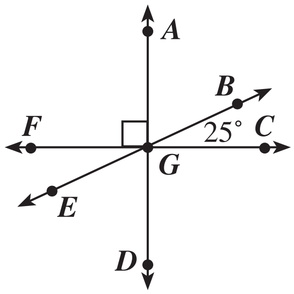
$m\angle JNL =$ __________ + __________

($m\angle JNK$) ($m\angle KNL$)

$=$ __________

• Angle Problem Solving •

1) $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ intersect at point G.

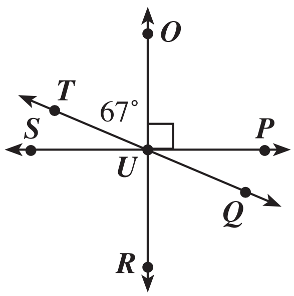


$m\angle AGB =$ ______, $m\angle DGE =$ ______

$m\angle CGE =$ __________ + __________
($m\angle CGD$) ($m\angle DGE$)

= __________

2) $\overleftrightarrow{OR}$, $\overleftrightarrow{PS}$, and $\overleftrightarrow{QT}$ intersect at point U.

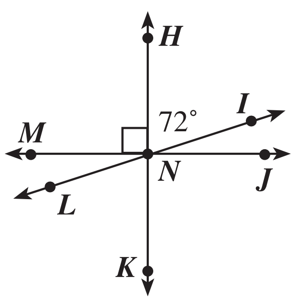


$m\angle SUT =$ ______, $m\angle PUQ =$ ______

$m\angle OUQ =$ __________ + __________
($m\angle OUP$) ($m\angle PUQ$)

= __________

3) $\overleftrightarrow{HK}$, $\overleftrightarrow{IL}$, and $\overleftrightarrow{JM}$ intersect at point N.

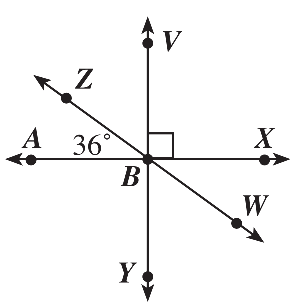


$m\angle INJ =$ __________

$m\angle INK =$ __________

$m\angle JNL =$ __________

4) $\overleftrightarrow{VY}$, $\overleftrightarrow{AX}$, and $\overleftrightarrow{WZ}$ intersect at point B.

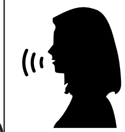


$m\angle ZBV =$ __________

$m\angle XBW =$ __________

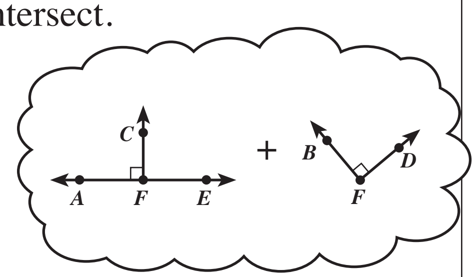
$m\angle ABW =$ __________

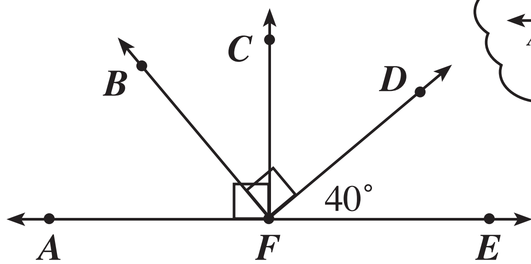
• Finding Missing Angles •

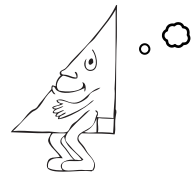


Let's look at a problem where the given right angles intersect.

Example: Given $\overleftrightarrow{AE}$, find $m\angle AFB$.



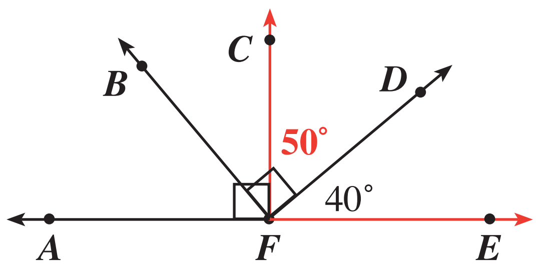




Since $\angle AFC$ and $\angle CFE$ form a linear pair, they are supplementary. We are given $m\angle AFC = 90°$, so $m\angle CFE = 180° - 90° = 90°$.

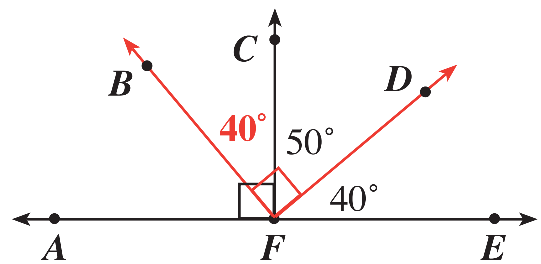
Since $\angle CFD$ and $\angle DFE$ are complementary:

$$m\angle CFD = 90° - 40° = 50°.$$



Since $\angle BFC$ and $\angle CFD$ are complementary:

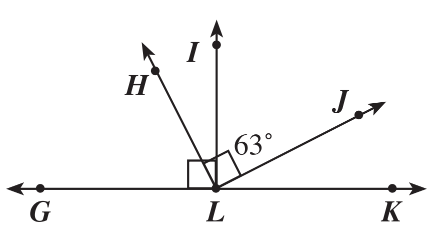
$$m\angle BFC = 90° - 50° = 40°.$$



Since $\angle AFB$ and $\angle BFC$ are complementary, $m\angle AFB = 90° - 40° = 50°$.

***Try this*:**

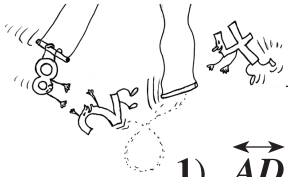
1) Given $\overleftrightarrow{GK}$, find the following angles.



$m\angle JLK =$ __________

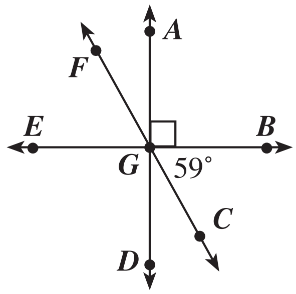
$m\angle HLI =$ __________

$m\angle GLH =$ __________



• Angle Problem Solving •

1) $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ intersect at point G.

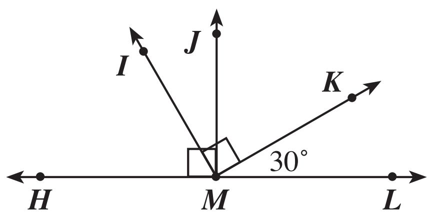


$m\angle AGF$ = ________

$m\angle CGE$ = ________

$m\angle DGF$ = ________

2) Given $\overleftrightarrow{HL}$, find the following angles.

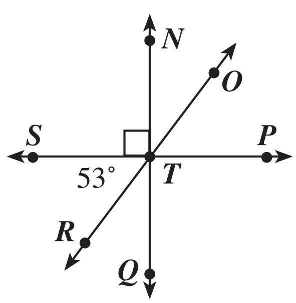


$m\angle JMK$ = ________

$m\angle IMJ$ = ________

$m\angle HMI$ = ________

3) $\overleftrightarrow{NQ}$, $\overleftrightarrow{OR}$, and $\overleftrightarrow{PS}$ intersect at point T.

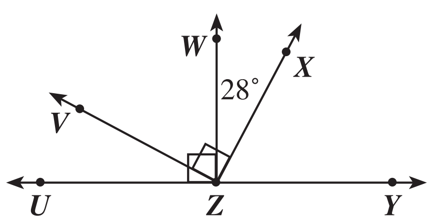


$m\angle NTO$ = ________

$m\angle OTQ$ = ________

$m\angle PTR$ = ________

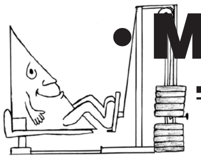
4) Given $\overleftrightarrow{UY}$, find the following angles.



$m\angle XZY$ = ________

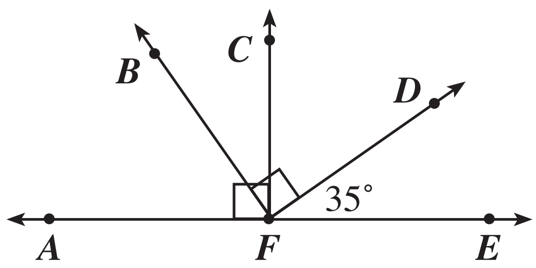
$m\angle UZV$ = ________

$m\angle VZY$ = ________



• Mastery Check: Angle Problem Solving • ☆

1) Given $\overleftrightarrow{AE}$, find the following angles.

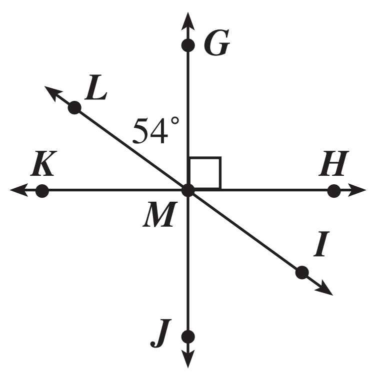


$m\angle CFD =$ ________

$m\angle BFC =$ ________

$m\angle AFB =$ ________

2) $\overleftrightarrow{GJ}$, $\overleftrightarrow{HK}$, and $\overleftrightarrow{IL}$ intersect at point ***M***.



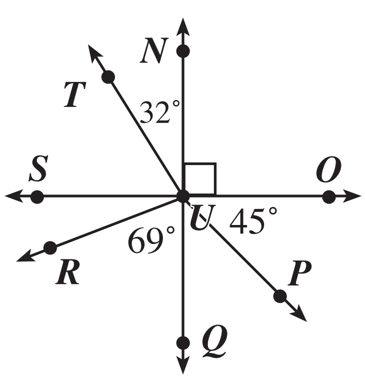
$m\angle IMJ =$ ________

$m\angle KML =$ ________

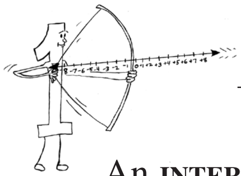
$m\angle GMI =$ ________

Challenge:

3) $\overleftrightarrow{NQ}$ and $\overleftrightarrow{OS}$ intersect at point ***U***.



$m\angle RUT + m\angle PUQ =$ ________



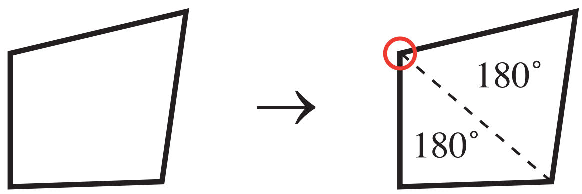
• Interior Angles of a Polygon •

Direct Teaching

An **INTERIOR ANGLE** of a polygon is an angle formed by two adjacent sides that lies *inside* the polygon. To determine the sum of the **INTERIOR ANGLES** of a polygon, we can divide it up into triangles.

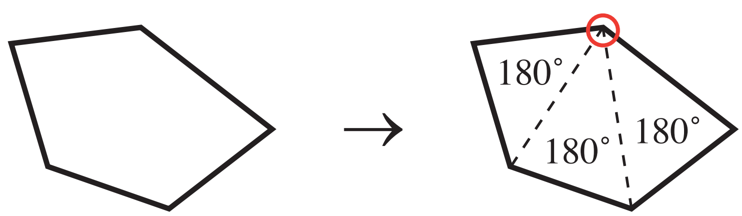
Since we already know that the measures of any triangle's three **INTERIOR ANGLES** have a sum of 180˚, we can count the number of triangles drawn from one vertex to determine the sum of the **INTERIOR ANGLES** for any *convex* polygon.

EXAMPLE 1: What is the sum of the interior angles of a quadrilateral?



A quadrilateral can be divided into two triangles by starting at any one vertex and drawing a diagonal to the other vertex. So, it has an **INTERIOR ANGLE** sum of 180˚ × 2 = 360˚.

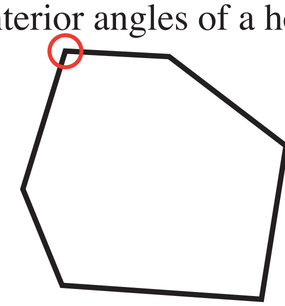
EXAMPLE 2: What is the sum of the interior angles of a pentagon?



A pentagon can be divided into three triangles by starting at any one vertex and drawing diagonals to each other vertex. So, it has an **INTERIOR ANGLE** sum of 180˚ × 3 = 540˚.

Try this:

1) Divide the hexagon below into triangles by drawing diagonals starting at the circled vertex. What is the sum of the interior angles of a hexagon?



180˚ × ________ = ___________

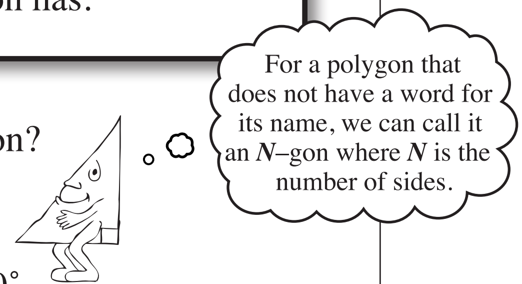
• Interior Angle Sum •

Direct Teaching

Instead of drawing and counting triangles to determine the sum of the INTERIOR ANGLES of a polygon, let's use the following formula.

> The sum of the INTERIOR ANGLES of a convex polygon = $180^\circ \times (N - 2)$
> where N is the number of sides the polygon has.

EXAMPLE: What is the sum of the interior angles of a 22–gon?



$$180^\circ \times (N - 2)$$
$$= 180^\circ \times (22 - 2) = 180^\circ \times 20 = 3{,}600^\circ$$

Try these:

1) What is the sum of the interior angles of an 11–gon?

$180^\circ \times ($ 11 $- 2)$ (blank labelled N)

= ___________

2) What is the sum of the interior angles of a 13–gon?

$180^\circ \times ($ ______ $- 2)$ (blank labelled N)

= ___________

3) What is the sum of the interior angles of a 9–gon?

____

4) What is the sum of the interior angles of a 14–gon?

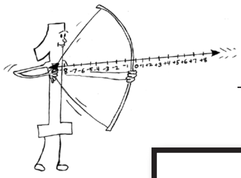
____

Extending Knowledge

5) A polygon has an interior angle sum of 18,000°. How many sides does it have?

$$180^\circ \times (N - 2) = 18{,}000^\circ$$

N = ___________



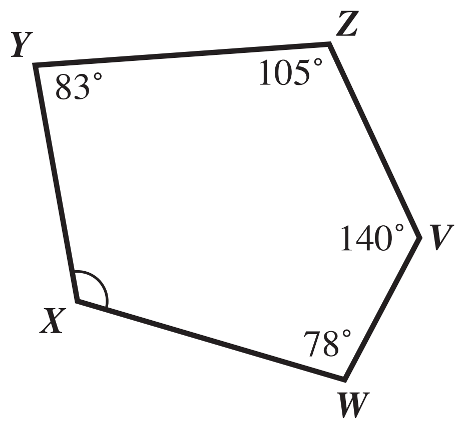
• Finding the Missing Angle •

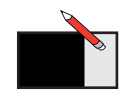
Direct Teaching

The sum of the INTERIOR ANGLES of a convex polygon = 180˚ × (*N* – 2) where *N* is the number of sides the polygon has.

Now that we have a formula to find the sum of the INTERIOR ANGLES of any convex polygon, let's look at a problem where we need to find a missing angle measurement of a polygon.

EXAMPLE: What is the measure of $\angle X$?





Whole MINUS Part

We can solve this problem by taking the whole (the sum of the angles) and subtracting the parts (the individual angles).

To find the whole, use the formula for the sum of the INTERIOR ANGLES of a convex polygon when $N = 5$.

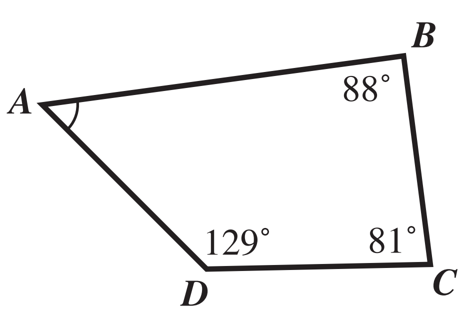
$$180^\circ \times (N - 2) = 180^\circ \times (5 - 2) = 180^\circ \times 3 = 540^\circ$$

Now that we know the sum of all the angles, we can subtract each known angle measure to find the unknown angle measure.

$$m\angle X = \text{WHOLE} - \text{SUM OF PARTS}$$
$$m\angle X = 540^\circ - (78^\circ + 140^\circ + 105^\circ + 83^\circ) = 540^\circ - 406^\circ$$
$$m\angle X = 134^\circ$$

***Try this*:**

1) What is the measure of $\angle A$?

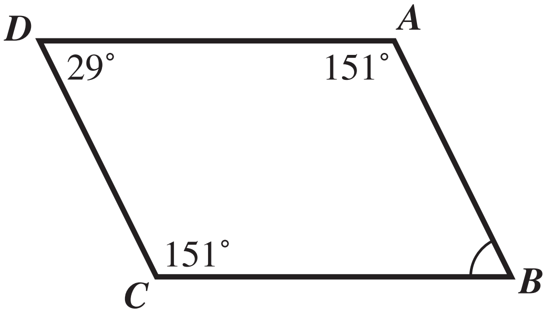


WHOLE = 180˚ × (______ – 2) = ____________
N

$m\angle A$ = ______ – ____________ = ____________
WHOLE SUM OF PARTS

• Finding the Missing Angle •

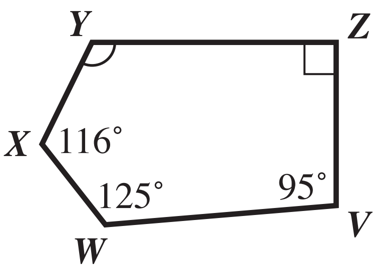
1) What is the measure of $\angle B$?



$m\angle B =$ 360° − ______
WHOLE SUM OF PARTS

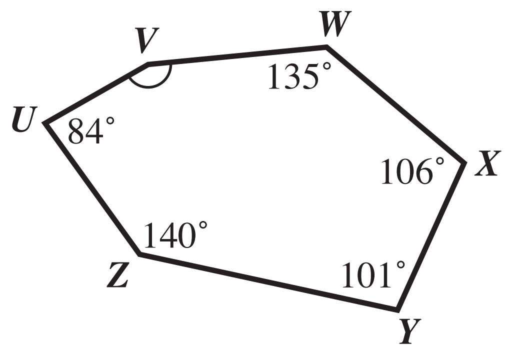
$m\angle B =$ ____________

2) What is the measure of $\angle Y$?



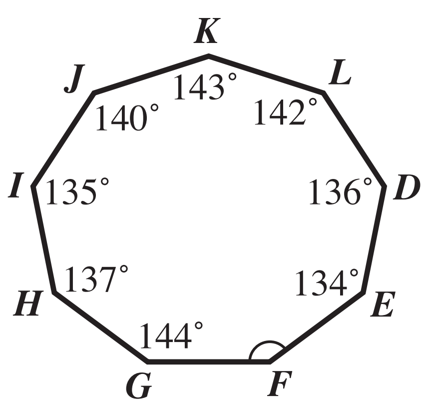
$m\angle Y =$ ____________

3) What is the measure of $\angle V$?



$m\angle V =$ ____________

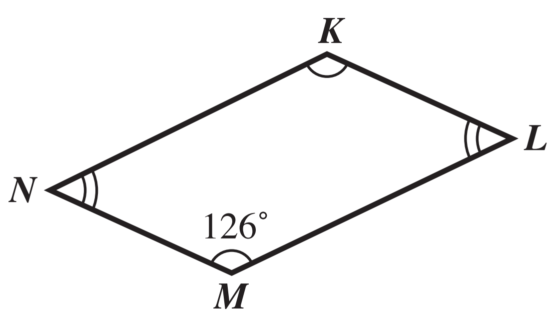
4) What is the measure of $\angle F$?



$m\angle F =$ ____________

5) What is the measure of $\angle L$?

Extending Knowledge



$m\angle L =$ ____________

• Angles of a Regular Polygon •

Direct Teaching

When we want to know the measure of a *single angle* of a **REGULAR POLYGON**, we find the sum of all the angles and then divide it by the number of angles, which is the same as the number of sides.

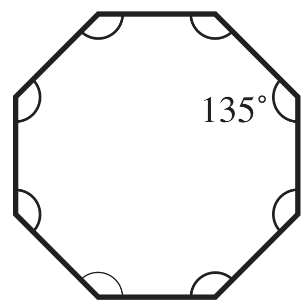
> The measure of each interior angle of a regular polygon = $\dfrac{180^\circ \times (\boldsymbol{N} - 2)}{\boldsymbol{N}}$
>
> where N is the number of sides the polygon has.

EXAMPLE: What is the measure of each angle of a regular octagon?

An octagon has 8 equal sides and 8 equal angles, so:

$$\frac{180^\circ \times (\boldsymbol{N} - 2)}{\boldsymbol{N}}$$

$$= \frac{180^\circ \times (8 - 2)}{8} = \frac{180^\circ \times 6}{8} = \frac{1{,}080^\circ}{8} = 135^\circ$$



The measure of each angle of a regular octagon is 135°.

Try these:

1) What is the measure of each angle of a regular pentagon?

____

2) What is the measure of each angle of a regular 15–gon?

____

• Interior Angles of a Polygon •

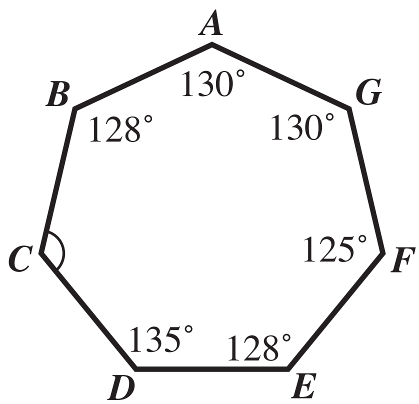
1) What is the sum of the interior angles of a 30–gon?

____

2) What is the sum of the interior angles of a nonagon (9–gon)?

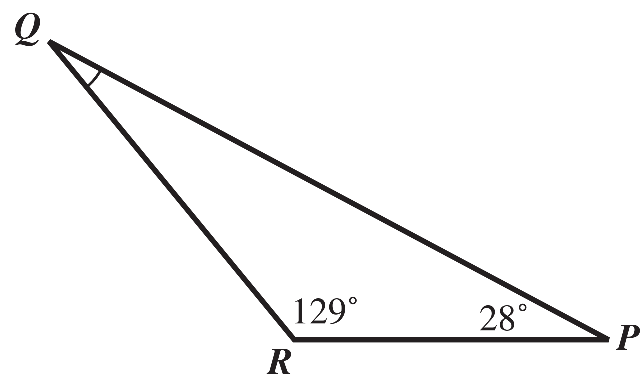
____

3) What is the measure of $\angle C$?



$m\angle C =$ __________

4) What is the measure of $\angle Q$?



$m\angle Q =$ __________

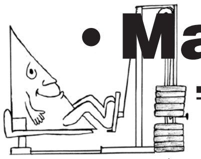
5) What is the measure of each angle of a regular 18–gon?

____

6) Each angle of a regular polygon is 60°. What shape is it?

____

Extending Knowledge

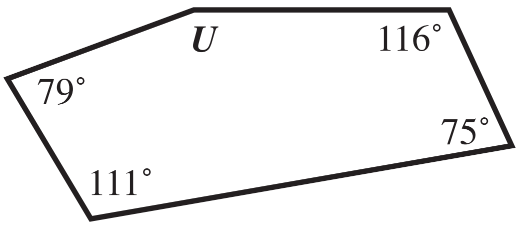


• Mastery Check: Interior Angles of a Polygon • ☆

1) What is the sum of the interior angles of an octagon?

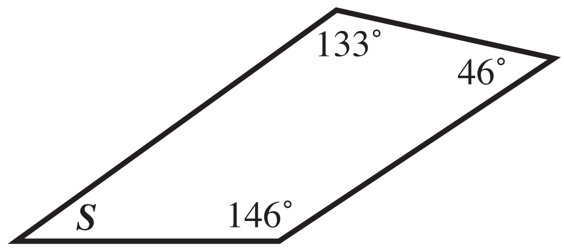
____

2) What is the measure of $\angle U$?



$m\angle U =$ __________

3) What is the measure of $\angle S$?



$m\angle S =$ __________

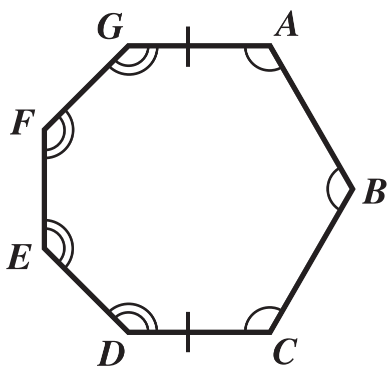
4) What is the measure of each interior angle of a regular decagon?

____

Challenge:

5) What are the measures of $\angle A$ and $\angle D$?

HINTS: Break the polygon into two parts. Note that $\overline{GA}$ and $\overline{CD}$ are parallel.



$m\angle A =$ __________

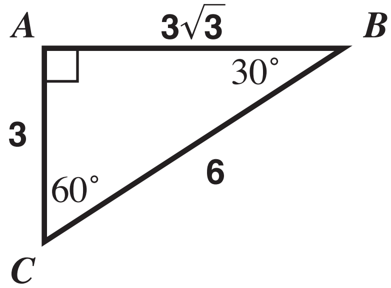
$m\angle D =$ __________

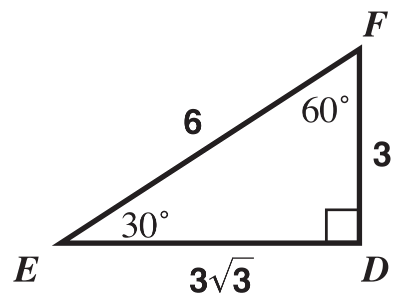
• Determining Congruence •

We can determine whether two polygons are congruent by examining their corresponding sides and their corresponding angles for congruence.

Direct Teaching

Example: Are triangles ***ABC*** and ***DEF*** congruent?





First let's show that the corresponding sides of each triangle are congruent:

$\overline{AB} \cong \overline{DE} = 3\sqrt{3}$ $\overline{BC} \cong \overline{EF} = 6$ $\overline{AC} \cong \overline{DF} = 3$

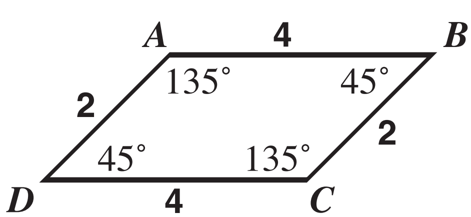
Then let's show that the corresponding angles of each triangle are congruent:

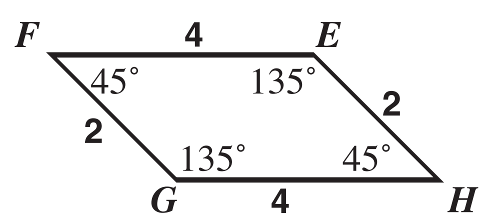
$\angle A \cong \angle D = 90^\circ$ $\angle B \cong \angle E = 30^\circ$ $\angle C \cong \angle F = 60^\circ$

Since their corresponding sides are congruent and their corresponding angles are congruent, triangles ***ABC*** and ***DEF*** are congruent.

Try this: **Fill in the blanks to determine whether the polygons are congruent.**

1) Are parallelograms ***ABCD*** and ***EFGH*** congruent?





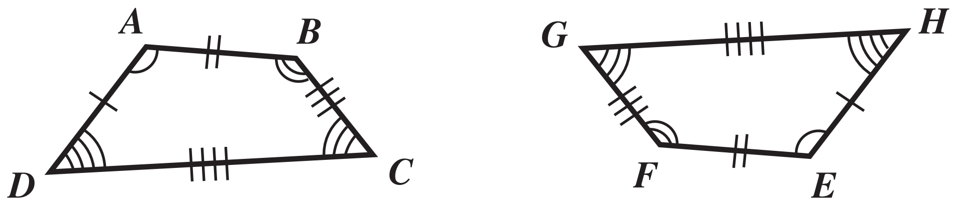
$\overline{AB} \cong$ ________, $\overline{BC} \cong$ ________, $\overline{CD} \cong$ ________, $\overline{AD} \cong$ ________

$\angle A \cong$ ________, $\angle B \cong$ ________, $\angle C \cong$ ________, $\angle D \cong$ ________

So, parallelograms ***ABCD*** and ***EFGH*** are congruent.

• Determining Congruence •

1) Are quadrilaterals ***ABCD*** and ***EFGH*** congruent?

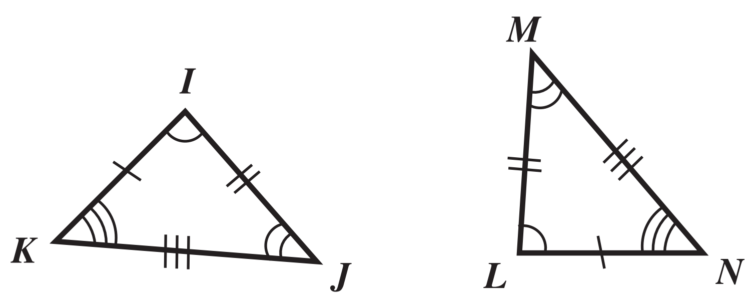


List the congruent corresponding side pairs.

List the congruent corresponding angle pairs.

So, quadrilaterals ***ABCD*** and ***EFGH*** are congruent.

2) Are triangles ***IJK*** and ***LMN*** congruent?



List the congruent corresponding side pairs.

List the congruent corresponding angle pairs.

So, triangles ***IJK*** and ***LMN*** are congruent.

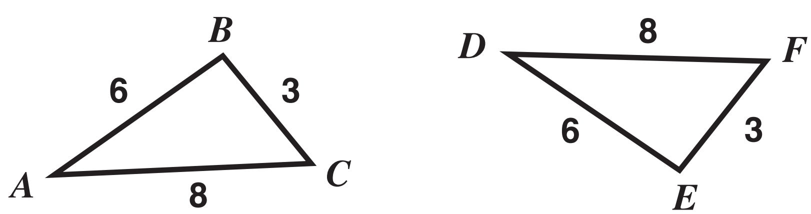
• Congruent Triangles: Side–Side–Side •

Direct Teaching

Triangles have a special property that allows us to prove they are congruent solely by their side lengths. This property is known as **TRIANGLE RIGIDITY**.

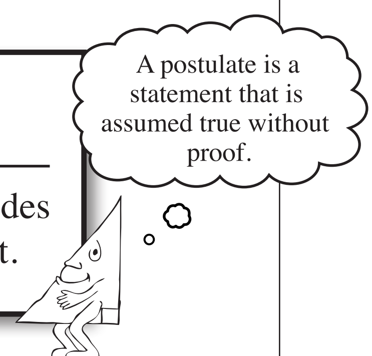
> **Triangle Rigidity**
>
> A triangle can only have one unique shape if its side lengths are fixed.

What this means is that if we are given three side lengths of a triangle, they will always create the same shape. Let's see if the following triangles are congruent.



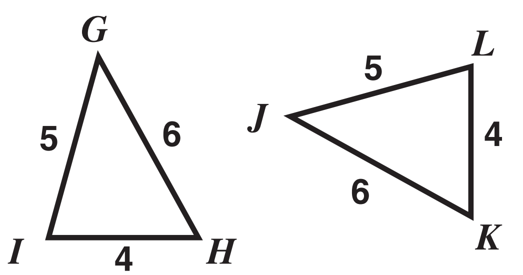
Since both triangles have side lengths 3, 6, and 8, by **TRIANGLE RIGIDITY** they must be the same shape. So, $\triangle ABC \cong \triangle DEF$. This leads us to the following postulate:

> **The Side–Side–Side Postulate (SSS)**
>
> If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.



***Try this*: Fill in the blanks.**

1)



$\overline{GH} \cong$ ________

$\overline{HI} \cong$ ________

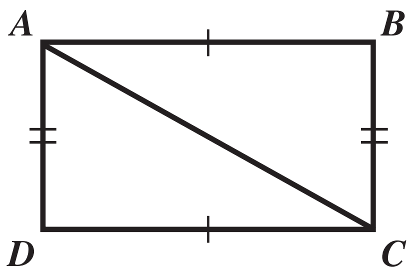
$\overline{GI} \cong$ ________

So, $\triangle GHI \cong \triangle JKL$ by SSS.

• Side–Side–Side •

We can use SSS to show that two triangles are congruent.

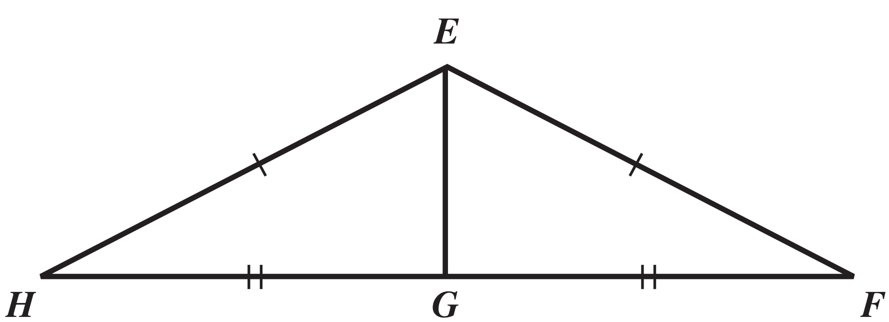
Example: Is $\triangle ABC$ congruent to $\triangle CDA$?



In order to show that $\triangle ABC$ is congruent to $\triangle CDA$, we must show that three sides of $\triangle ABC$ are congruent to three sides of $\triangle CDA$.

1) $\overline{AB} \cong \overline{DC}$	We are given this from the diagram.
2) $\overline{BC} \cong \overline{AD}$	We are given this from the diagram.
3) $\overline{AC} \cong \overline{AC}$	By the reflexive property, we know that any side is congruent to itself.
4) $\triangle ABC \cong \triangle CDA$	So, by SSS, we determine that $\triangle ABC$ is congruent to $\triangle CDA$.

***Try this*: Figure may not be drawn to scale.**



1) Is $\triangle EFG$ congruent to $\triangle EHG$? Explain:

____

____

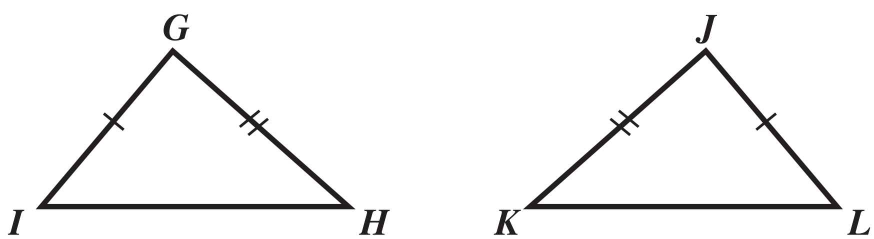
• Side–Side–Side •

Figures may not be drawn to scale.

1) Triangle ***ABC*** has side lengths of 7 cm, 4 cm, and 5 cm. What are the side lengths of the congruent triangle ***DEF***?

____

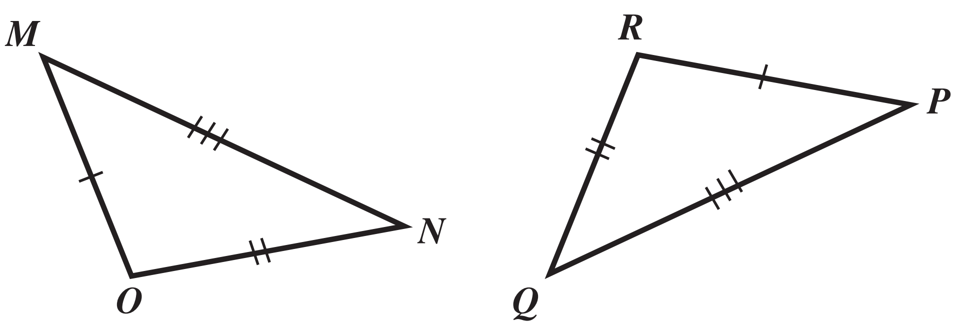
____



2) Is there enough information to determine whether $\triangle$***GHI*** is congruent to $\triangle$***JKL***? If not, what additional information would you need?

____

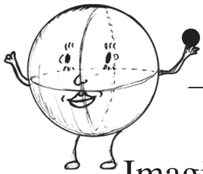
____



3) Is $\triangle$***MNO*** congruent to $\triangle$***PQR***? Explain:

____

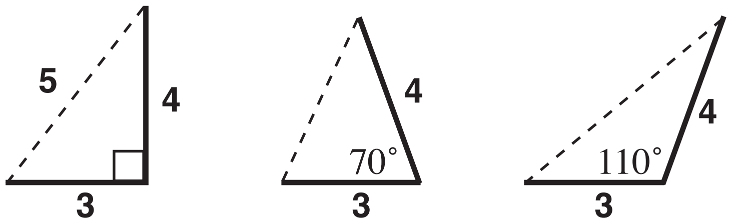
____

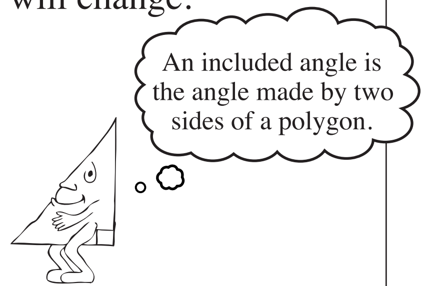


• Congruent Triangles: Side–Angle–Side •

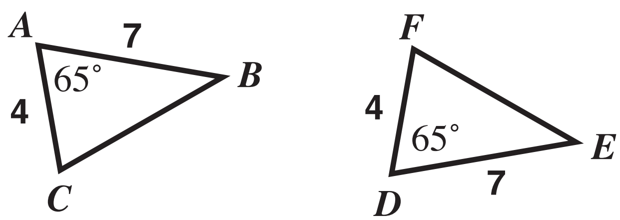
Direct Teaching

Imagine we are given two sides of a triangle with lengths 3 and 4 with an included angle of measure 90°. By the Pythagorean Theorem, the third side will always be of length 5. If we increase or decrease the angle, the third side will change.





If we are given the lengths of two sides of a triangle and an included angle, the third side will always be a fixed length. Let's see if the following triangles are congruent.



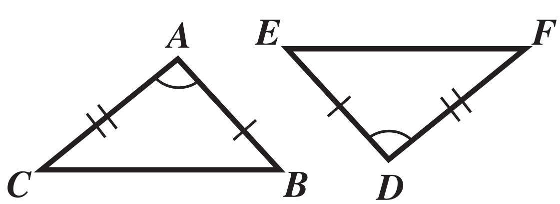
Since both triangles have side lengths 4 and 7 and an included angle of measure 65°, the third side of both triangles must be the same, fixed length. So, since both triangles have the same side lengths, they are congruent triangles. This leads us to the following postulate:

The Side–Angle–Side Postulate (SAS)

If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

***Try this*: Fill in the blanks.**

1)



$\overline{AB} \cong$ ________

$\angle A \cong$ ________

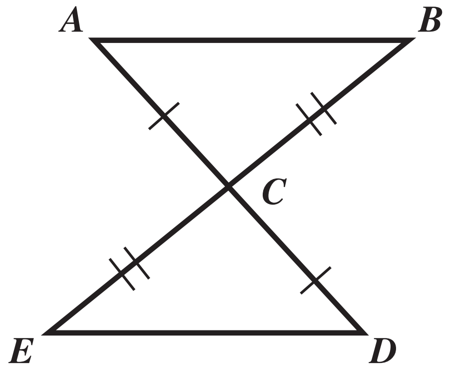
$\overline{AC} \cong$ ________

So, $\triangle ABC \cong \triangle DEF$ by SAS.

• Side–Angle–Side •

We can use SAS to prove triangles are congruent.

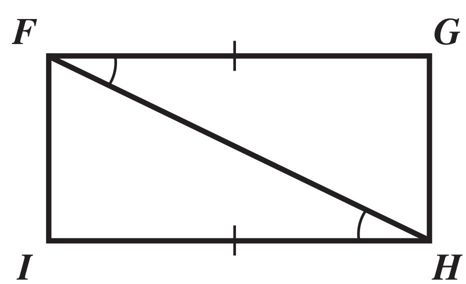
Example: Is $\triangle ABC$ congruent to $\triangle EDC$? $\overline{AD}$ and $\overline{BE}$ are line segments.



In order to show that $\triangle ABC$ is congruent to $\triangle EDC$, we must show that two sides and the included angle of $\triangle ABC$ are congruent to two sides and the included angle of $\triangle EDC$.

1) $\overline{AC} \cong \overline{CD}$	We are given this from the diagram.
2) $\angle ACB \cong \angle DCE$	Vertical angles are congruent.
3) $\overline{BC} \cong \overline{CE}$	We are given this from the diagram.
4) $\triangle ABC \cong \triangle EDC$	So, by SAS, we determine that $\triangle ABC$ is congruent to $\triangle EDC$.

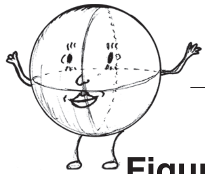
Try this: **Figure may not be drawn to scale.**



1) Is $\triangle FGH$ congruent to $\triangle HIF$? Explain:

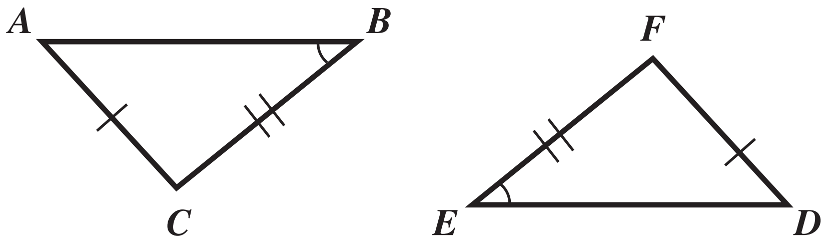
____

____



• Side–Angle–Side •

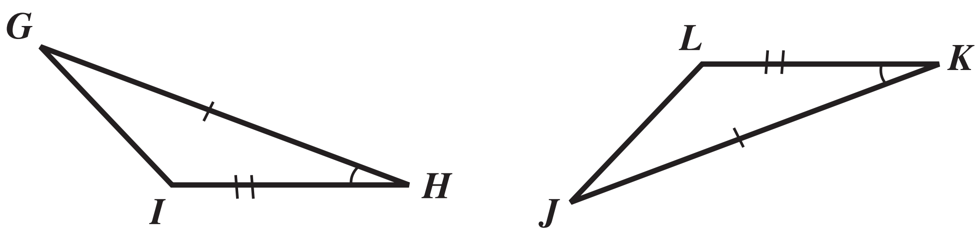
Figures may not be drawn to scale.



1) Is there enough information to determine whether $\triangle \boldsymbol{ABC}$ is congruent to $\triangle \boldsymbol{DEF}$ by SAS? If not, what additional information would you need?

____

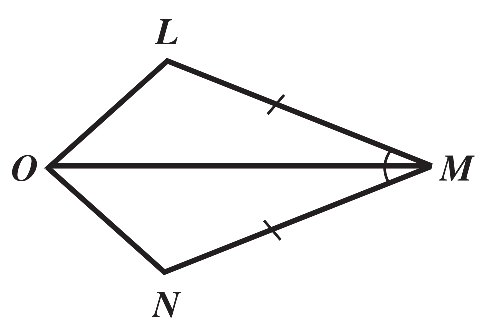
____



2) Is $\triangle \boldsymbol{IGH}$ congruent to $\triangle \boldsymbol{LJK}$? Explain:

____

____



3) Is $\triangle \boldsymbol{LMO}$ congruent to $\triangle \boldsymbol{NMO}$? Explain:

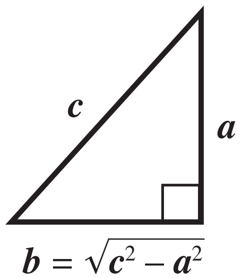
____

____

• Hypotenuse–Leg •

Direct Teaching

When we have a right triangle, if we are given the lengths of its hypotenuse and a leg, we can find the length of its other leg using the Pythagorean Theorem.



Since $\boldsymbol{a}^2 + \boldsymbol{b}^2 = \boldsymbol{c}^2$*, given a right triangle with hypotenuse* **c** *and leg* **a***, the third side will always be* $\sqrt{\boldsymbol{c}^2 - \boldsymbol{a}^2}$.

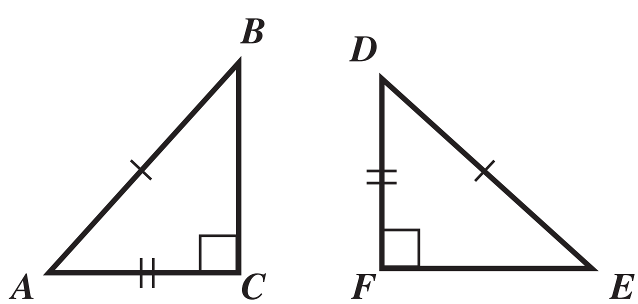
So, if we are given the lengths of the hypotenuse and a leg of a right triangle, the length of its third side is fixed. It follows that if the hypotenuse and a leg of a right triangle are congruent to the hypotenuse and a leg of another right triangle, their third sides will also be congruent. By SSS, the triangles are congruent. This leads us to the following postulate:

The Hypotenuse–Leg Postulate (HL)

If the hypotenuse and a leg of a right triangle are congruent to the hypotenuse and a leg of another right triangle, the triangles are congruent.

Try this:

1)



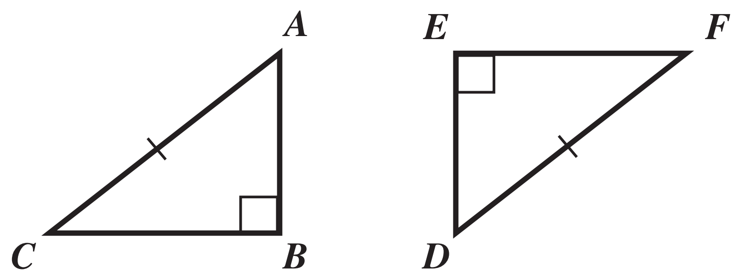
Is $\triangle ABC$ congruent to $\triangle DEF$? Explain:

____

____

• Congruent Triangles •

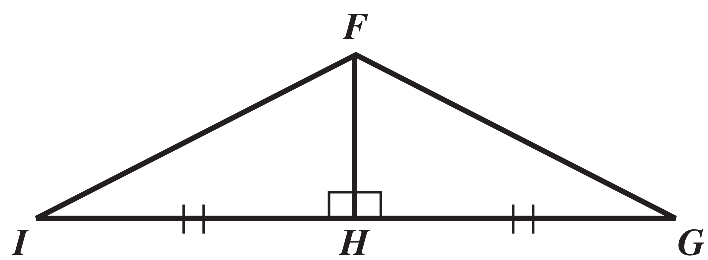
Figures may not be drawn to scale.



1) Is there enough information to determine whether $\triangle \boldsymbol{ABC}$ is congruent to $\triangle \boldsymbol{DEF}$? If not, what additional information would you need?

____

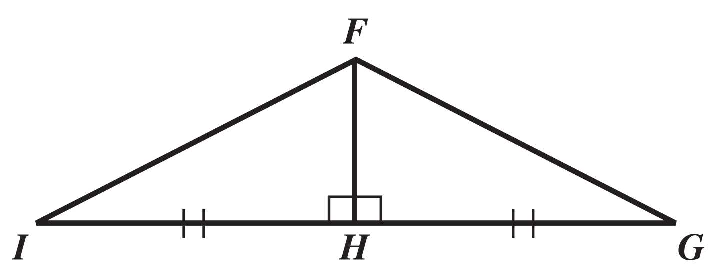
____



2) Is $\triangle \boldsymbol{FGH}$ congruent to $\triangle \boldsymbol{FIH}$? Explain:

____

____



3) How can you show $\triangle \boldsymbol{FGH}$ is congruent to $\triangle \boldsymbol{FIH}$ by SSS?

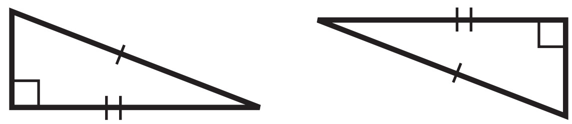
____

____

• Congruent Triangles •

Name the congruence postulate that can be used to conclude the pair of triangles are congruent. If there is not enough information, write "none."

1)



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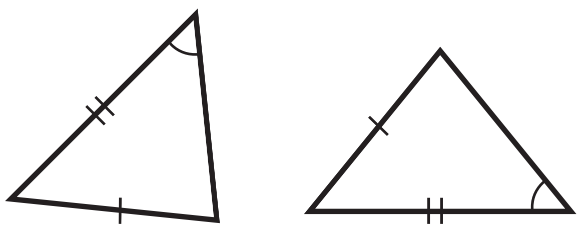
2)



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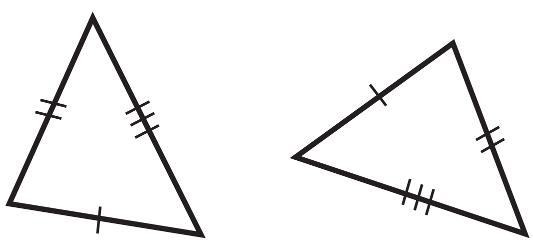
3)



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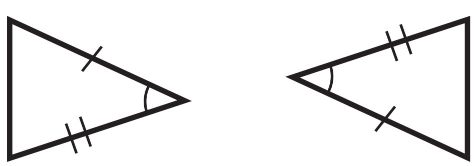
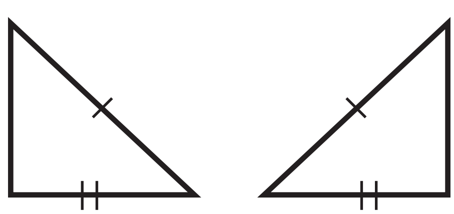
4)



____

5)



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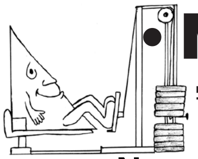
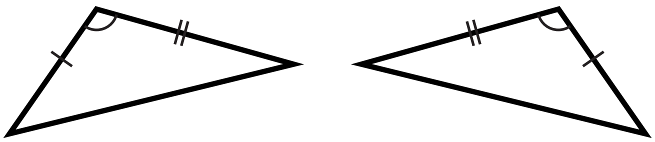


• Mastery Check: Congruent Triangles •

Name the congruence postulate that can be used to conclude the pair of triangles are congruent.

1)



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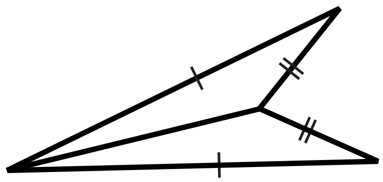
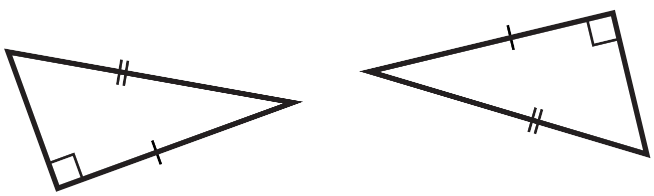
2)



____

3)



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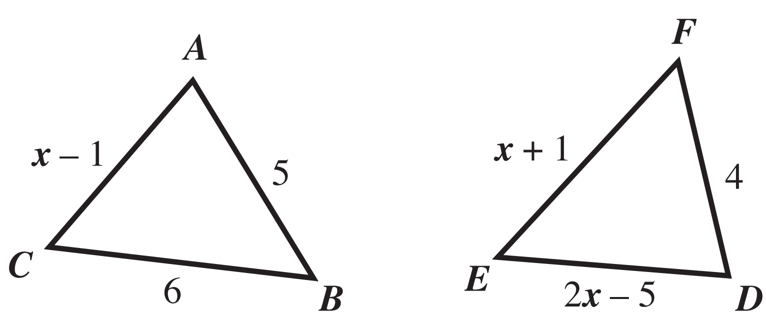
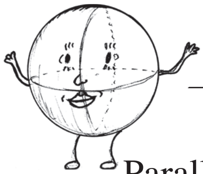
Challenge:

4) What value of x will make $\triangle ABC$ congruent to $\triangle DEF$?



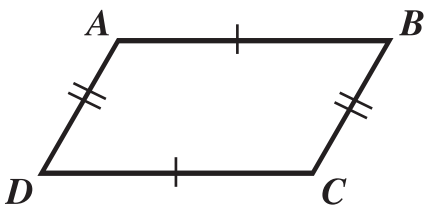
When $x =$ ________, $\triangle ABC$ is congruent to $\triangle DEF$.



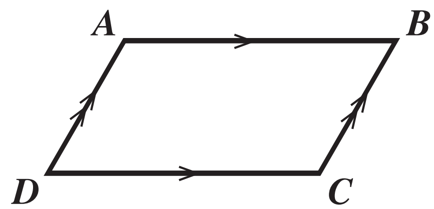
• Properties of Parallelograms •

Direct Teaching

Parallelograms are quadrilaterals with opposite sides that are parallel and congruent.



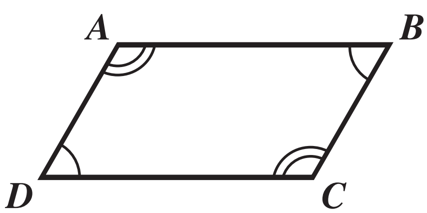
$\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$.



$\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$.

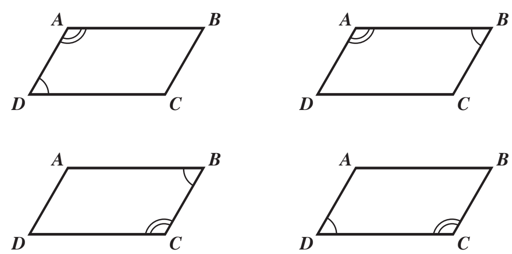
Parallelograms also have properties involving their angles.

Opposite angles of parallelograms are *congruent*.



$\angle A \cong \angle C$ and $\angle B \cong \angle D$.

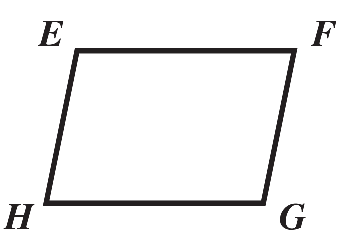
Consecutive angles of parallelograms are *supplementary*.



$\angle A$ & $\angle D$, $\angle A$ & $\angle B$, $\angle B$ & $\angle C$, and $\angle C$ & $\angle D$ are supplementary.

Try this: **Use parallelogram *EFGH* to answer the following.**

1)



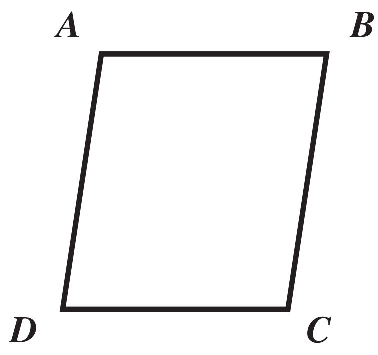
$EH =$ __________ $\overline{GH} \parallel$ __________

$\angle E \cong$ __________ $\angle F \cong$ __________

List the pairs of angles that are supplementary. ______________________

• Properties of Parallelograms •

1) Mark the congruent angles of parallelogram ***ABCD*** with the same number of arcs. Also, mark the congruent sides of parallelogram ***ABCD*** with the same number of ticks.

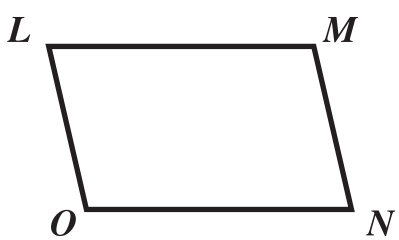


List the pairs of angles that are supplementary. ______________________

List the pairs of sides that are parallel. ______________________

Use parallelogram *LMNO* to answer the following.

2)



LM = __________ $\overline{LO}$ ∥ __________

∠***O*** ≅ __________ ∠***L*** ≅ __________

∠***L*** is supplementary to which two angles? _________________

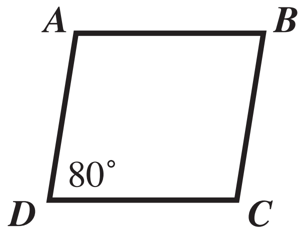
∠***M*** is supplementary to which two angles? _________________

∠***N*** is supplementary to which two angles? _________________

• Properties of Parallelograms •

We can use the properties of parallelograms to find the measures of their angles.

EXAMPLE: What are the measures of $\angle A$ and $\angle B$ in parallelogram ***ABCD***?



Since consecutive angles of parallelograms are supplementary:

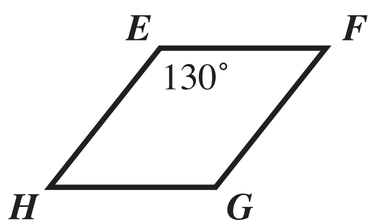
$$m\angle A + m\angle D = 180^\circ \Rightarrow m\angle A + 80^\circ = 180^\circ \Rightarrow m\angle A = 100^\circ$$

Since opposite angles of parallelograms are congruent:

$$\angle B \cong \angle D \Rightarrow m\angle B = m\angle D \Rightarrow m\angle B = 80^\circ$$

Try these:

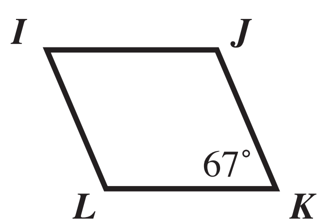
1)



EFGH is a parallelogram.

$m\angle G =$ ___________ $m\angle H =$ ___________

2)

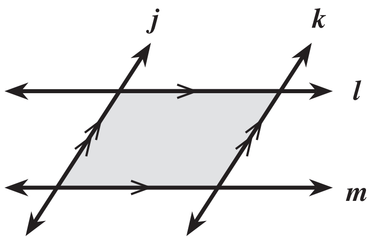


IJKL is a parallelogram.

$m\angle J =$ ___________ $m\angle I =$ ___________

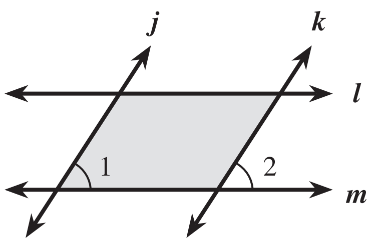
• Properties of Parallelograms •

Let's see why the opposite angles of a parallelogram are congruent.

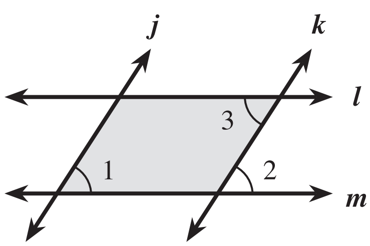


A parallelogram is made of two pairs of parallel line segments. Let's extend those line segments.

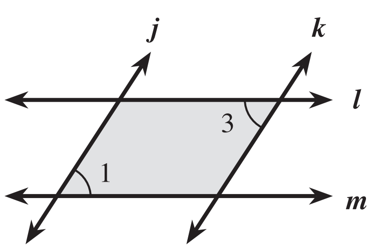
Here, $l \parallel m$ and $j \parallel k$.



Since $j \parallel k$ and is intersected by transversal m, by the Corresponding Angles Theorem, $\angle 1 \cong \angle 2$.



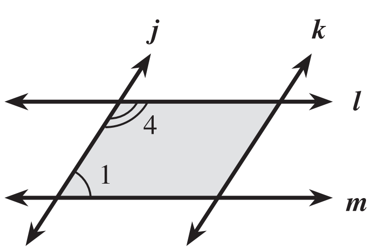
Since $l \parallel m$ and is intersected by transversal k, by the Alternate Interior Angles Theorem, $\angle 2 \cong \angle 3$.



We found that $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$. So, by the Transitive Property of Congruency, $\angle 1 \cong \angle 3$.

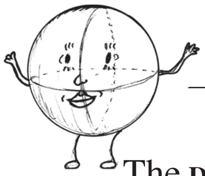
We can conclude that opposite angles of any parallelogram are congruent.

Try this:



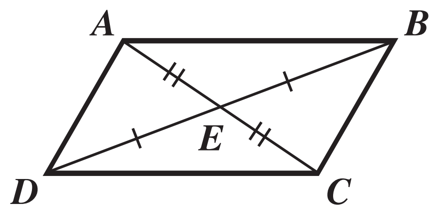
What theorem can we use to prove consecutive angles of a parallelogram are supplementary?

____



• Diagonals of Parallelograms •

The DIAGONALS of a parallelogram are the lines that connect its opposite vertices. The DIAGONALS of a parallelogram *bisect* each other.



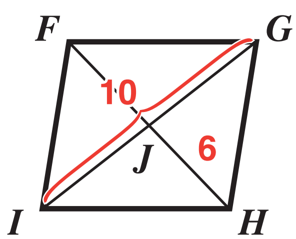
Here, $\overline{AC}$ and $\overline{BD}$ are the DIAGONALS of parallelogram $ABCD$.

$$AE = CE \Rightarrow AE = \tfrac{1}{2}AC \text{ and } CE = \tfrac{1}{2}AC.$$

$$BE = DE \Rightarrow BE = \tfrac{1}{2}BD \text{ and } DE = \tfrac{1}{2}BD.$$

EXAMPLE: $\overline{FH}$ and $\overline{GI}$ are diagonals of parallelogram $FGHI$. $FH = 12$ and $GJ = 5$.

What are the lengths of $\overline{HJ}$ and $\overline{GI}$?



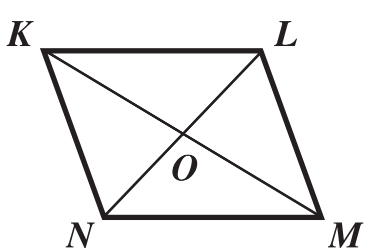
Since the diagonals of a parallelogram bisect each other:

$$HJ = \tfrac{1}{2}FH \Rightarrow HJ = \tfrac{1}{2}(12) = 6$$

$$GJ = \tfrac{1}{2}GI \Rightarrow 5 = \tfrac{1}{2}(GI) \Rightarrow GI = 10$$

Try this:

1) $\overline{KM}$ and $\overline{LN}$ are diagonals of parallelogram $KLMN$. $KM = 20$ and $NO = 9$.



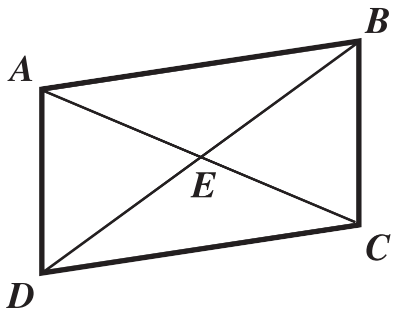
$KO =$ ___________

$LN =$ ___________

• Properties of Parallelograms •

1) $\overline{AC}$ and $\overline{BD}$ are diagonals of parallelogram $ABCD$.

$AC = 24$, $DE = 10$, $AD = 9$, and $m\angle ABC = 78^\circ$.



$CE =$ __________

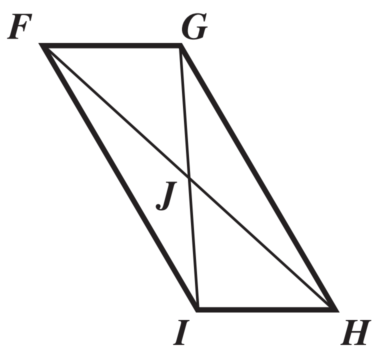
$BC =$ __________

$m\angle BCD =$ __________

$m\angle ADC =$ __________

2) $\overline{FH}$ and $\overline{GI}$ are diagonals of parallelogram $FGHI$.

$FG = 11$, $GJ = 17$, $FH = 37$, and $m\angle FIH = 121^\circ$.



$GI =$ __________

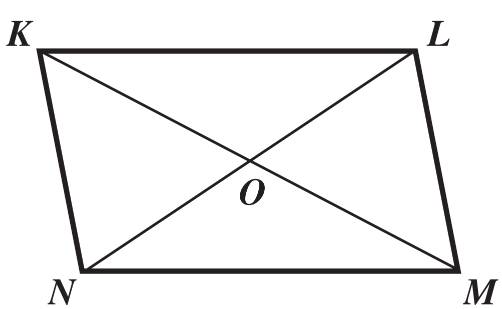
$FJ =$ __________

$IJ =$ __________

$m\angle FGH =$ __________

3) $\overline{KM}$ and $\overline{LN}$ are diagonals of parallelogram $KLMN$.

$LN = 4x$, $KO = 6y$, and $m\angle KNM = (4 + z)^\circ$.

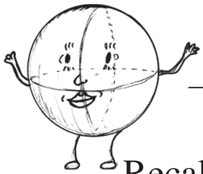


$KM =$ __________

$NO =$ __________

$m\angle NML =$ __________

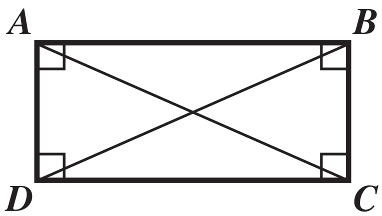
$m\angle KLM =$ __________



• Properties of Rectangles •

Recall that there are a number of special parallelograms. The diagonals of these special parallelograms have certain properties as well.

A rectangle is a parallelogram with four right angles. The diagonals of a rectangle are congruent.

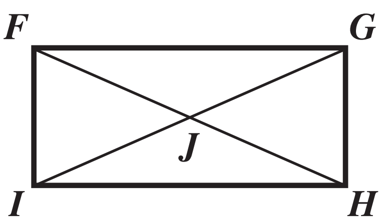


The diagonals of rectangle ***ABCD*** are congruent.

So, $\overline{AC} \cong \overline{BD}$.

Example: $\overline{FH}$ and $\overline{GI}$ are diagonals of rectangle ***FGHI***. $FH = 15$.

What are the lengths of $\overline{GI}$ and $\overline{HJ}$?



Since the diagonals of a rectangle are congruent:

$$FH = GI \Rightarrow GI = 15$$

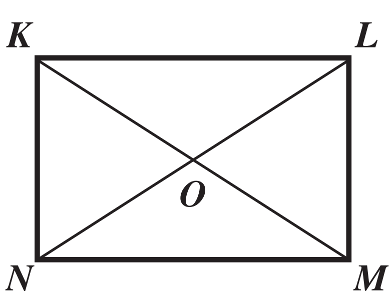
Since the diagonals of a parallelogram bisect each other:

$$HJ = \tfrac{1}{2}FH \Rightarrow HJ = \tfrac{1}{2}(15) = 7.5$$

Try this:

1) $\overline{KM}$ and $\overline{LN}$ are diagonals of rectangle ***KLMN***.

$LN = 23$.

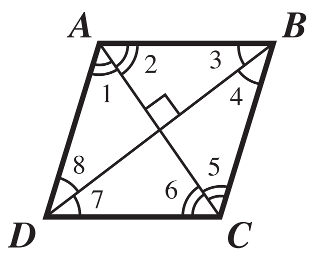


$MO =$ __________

$KM =$ __________

• Properties of Rhombuses •

A rhombus is a parallelogram with four sides the same length. The diagonals of a rhombus are perpendicular and bisect a pair of opposite angles.



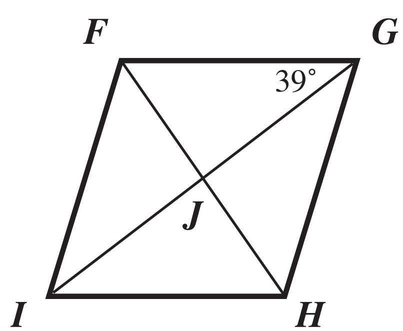
The diagonals of rhombus ***ABCD*** are perpendicular and bisect each angle.

So, $\overline{AC}$ is perpendicular to $\overline{BD}$ and

$\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$, $\angle 5 \cong \angle 6$, and $\angle 7 \cong \angle 8$.

Example: $\overline{FH}$ and $\overline{GI}$ are diagonals of rhombus ***FGHI***.

What is the measure of $\angle GFJ$?



Since the diagonals of a rhombus are perpendicular, $m\angle FJG = 90^\circ$.

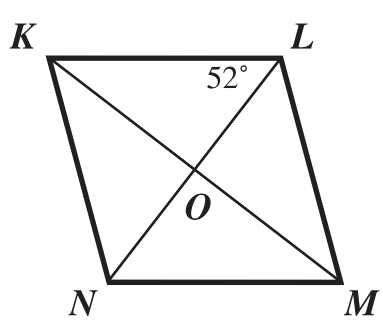
Since the interior angles of a triangle add up to 180˚,

$m\angle FJG + m\angle FGJ + m\angle GFJ = 180^\circ$.

So, $90^\circ + 39^\circ + m\angle GFJ = 180^\circ \Rightarrow m\angle GFJ = 51^\circ$.

Try this:

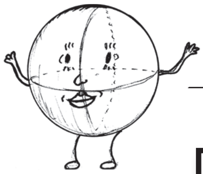
1) $\overline{KM}$ and $\overline{LN}$ are diagonals of rhombus ***KLMN***.



$m\angle OLM$ = __________

$m\angle LMO$ = __________

$m\angle OKN$ = __________



• Properties of Parallelograms •

Properties of Parallelograms

Opposite angles of parallelograms are congruent.

The diagonals of a parallelogram bisect each other.

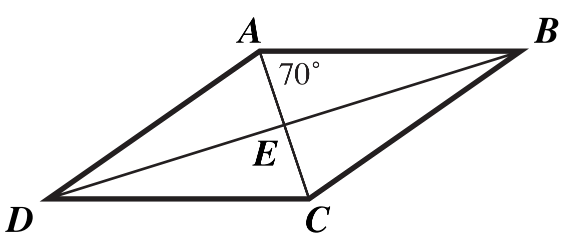
The diagonals of a rectangle are congruent.

The diagonals of a rhombus are perpendicular and bisect a pair of opposite angles.

---

1) $\overline{AC}$ and $\overline{BD}$ are diagonals of rhombus $ABCD$.

$AB = 16$, $CE = 5$, and $m\angle CAB = 70^\circ$.



$AC =$ ___________

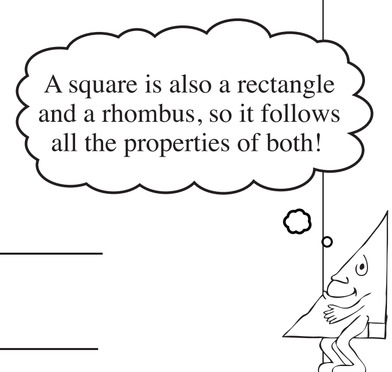
$BC =$ ___________

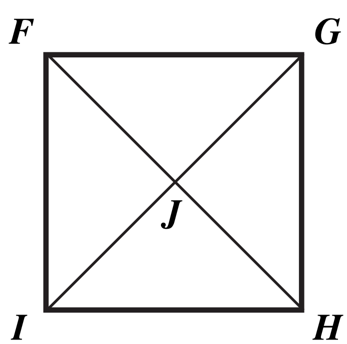
$m\angle DAC =$ ___________

$m\angle ABE =$ ___________

2) $\overline{FH}$ and $\overline{GI}$ are diagonals of square $FGHI$.

$FJ = 14$.





$FH =$ ___________

$GJ =$ ___________

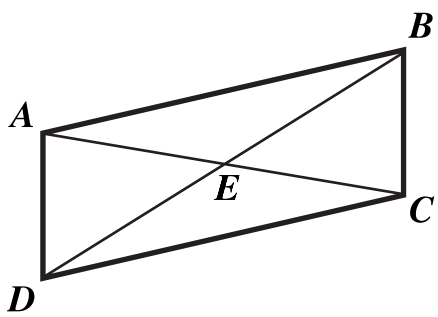
$m\angle FJG =$ ___________

$m\angle GFJ =$ ___________

• Properties of Parallelograms •

1) $\overline{AC}$ and $\overline{BD}$ are diagonals of parallelogram $ABCD$.

$AC = 18$, $BE = 12$, $m\angle ACD = 23°$, and $m\angle ACB = 80°$.



$AE =$ ___________

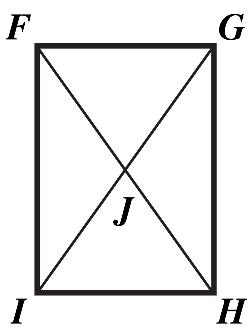
$BD =$ ___________

$m\angle BAC =$ ___________

$m\angle ADC =$ ___________

2) $\overline{FH}$ and $\overline{GI}$ are diagonals of rectangle $FGHI$.

$GI = 17$ and $m\angle FGI = 54°$.



$GJ =$ ___________

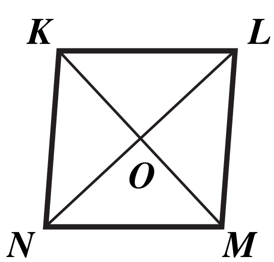
$FH =$ ___________

$m\angle IGH =$ ___________

$m\angle FIG =$ ___________

3) $\overline{KM}$ and $\overline{LN}$ are diagonals of rhombus $KLMN$.

$KM = 34$, $LN = 36$, and $m\angle KMN = 47°$.

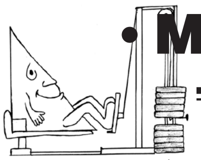


$NO =$ ___________

$m\angle NLM =$ ___________

$m\angle NOM =$ ___________

$m\angle KML =$ ___________

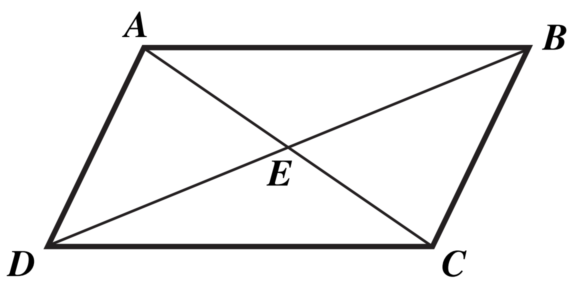


• Mastery Check: Properties of Parallelograms •

☆

1) $\overline{AC}$ and $\overline{BD}$ are diagonals of parallelogram $ABCD$.

$AC = 10$, $BE = 8$, and $m\angle BCD = 116^\circ$.



$AE =$ ___________

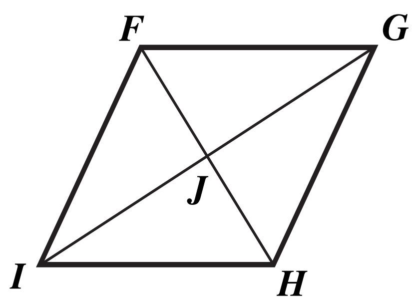
$BD =$ ___________

$m\angle ABC =$ ___________

$m\angle BAD =$ ___________

2) $\overline{FH}$ and $\overline{GI}$ are diagonals of rhombus $FGHI$.

$GI = 24$, $FJ = 9$, and $m\angle IFJ = 56^\circ$.



$IJ =$ ___________

$FH =$ ___________

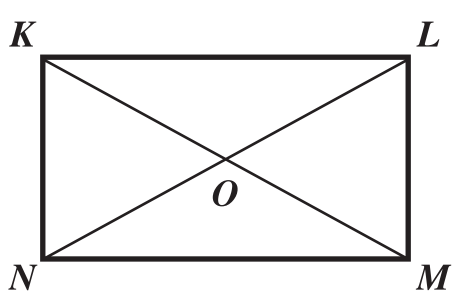
$m\angle GFJ =$ ___________

$m\angle GJH =$ ___________

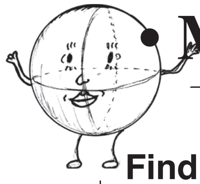
Challenge:

3) $\overline{KM}$ and $\overline{LN}$ are diagonals of rectangle $KLMN$.

$LM = 12$ m and $LO = 10$ m. What is the area of $KLMN$?

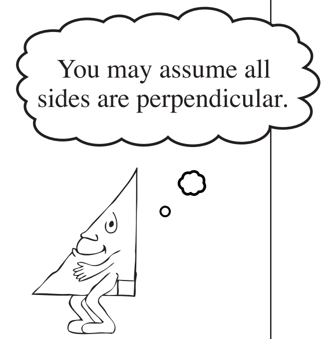


AREA = _______________

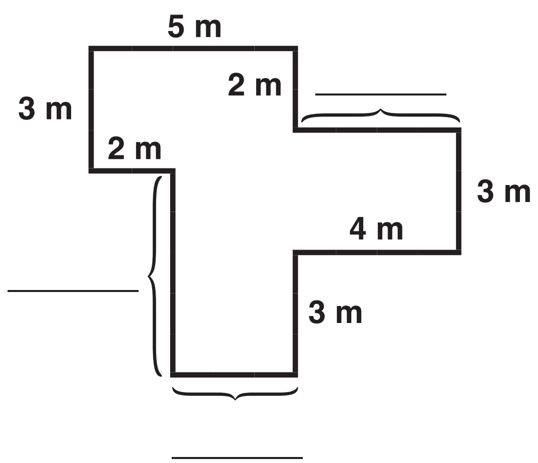


• Missing Dimensions of Composite Figures • ☆

Find the perimeters of each figure.

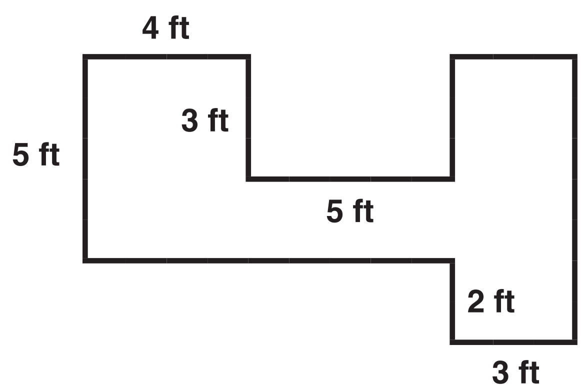


1)



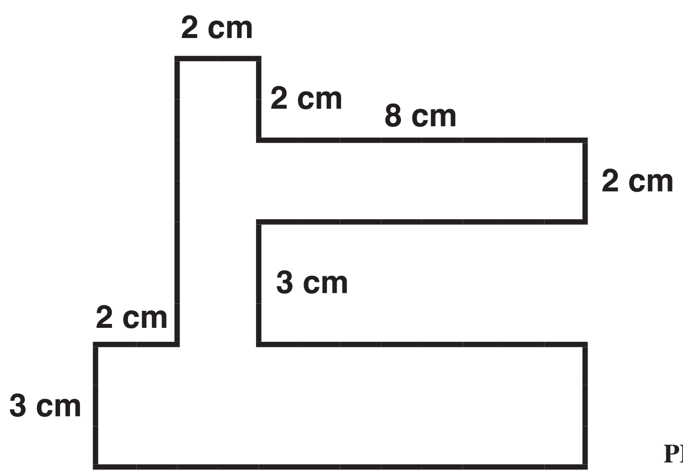
PERIMETER = ________________

2)



PERIMETER = ________________

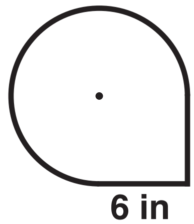
3)



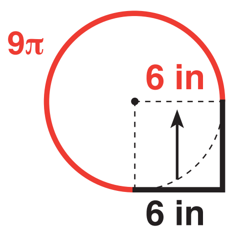
PERIMETER = ________________

• Perimeter of Composite Figures •

EXAMPLE: Find the perimeter of the following figure.

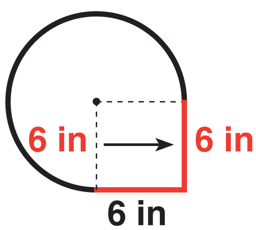


We can imagine the figure as a three–quarter circle and a square. To find the perimeter of the three–quarter circle, we will first find the circumference of the whole circle with the same radius.



We can figure out the radius of the circle from the bottom edge of the square (6 in). So, the circumference of the whole circle is $2\pi(6) = 12\pi$ in.

The length of the curved part of the figure is three–quarters of the circumference, which is $\frac{3}{4}(12\pi \text{ in}) = 9\pi$ in.

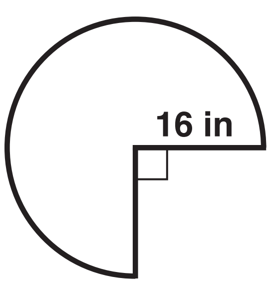


We can then add the lengths of the two sides of the square. So the perimeter of the whole figure is

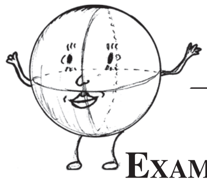
$$9\pi + 2(6) = (12 + 9\pi) \text{ in}.$$

***Try this*:** **Find the perimeter of the following figure. Leave answer in exact form.**

1)

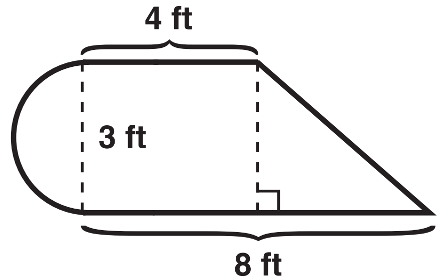


PERIMETER = ______________________

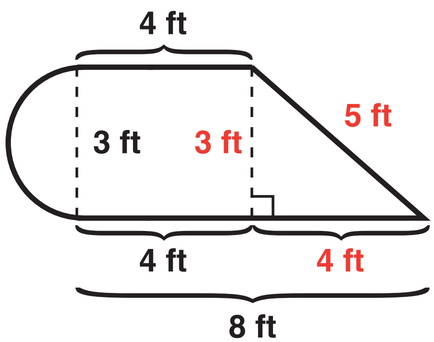


• Perimeter of Composite Figures •

EXAMPLE: Find the perimeter of the following figure.

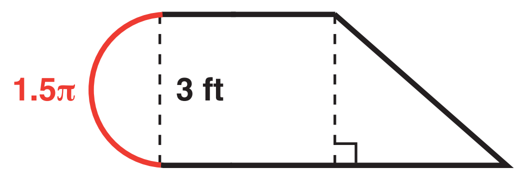


First find the missing dimensions.



Since the triangle part of the figure is a right triangle, we can use the **Pythagorean Theorem** to find the length of the hypotenuse.

$3^2 + 4^2 = c^2, \; c = 5$ ft



We can find the perimeter of the semicircle part of the figure by finding half the circumference of a circle with diameter of 3 feet.

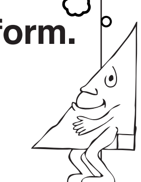
$\frac{1}{2} \bullet \pi \bullet d = \frac{1}{2} \bullet \pi \bullet 3 = 1.5\pi$ ft

Adding up all the sides of the figure, we find that the perimeter is:

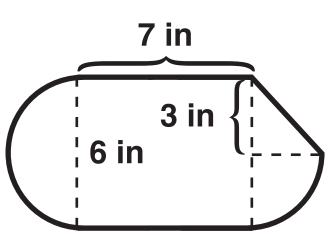
$4 + 5 + 8 + 1.5\pi = (17 + 1.5\pi)$ ft

You may assume all angles that look right are right angles.

Try this: **Find the perimeter of the following figure. Leave answer in exact form.**



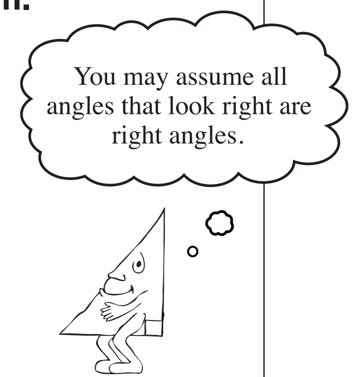
1)



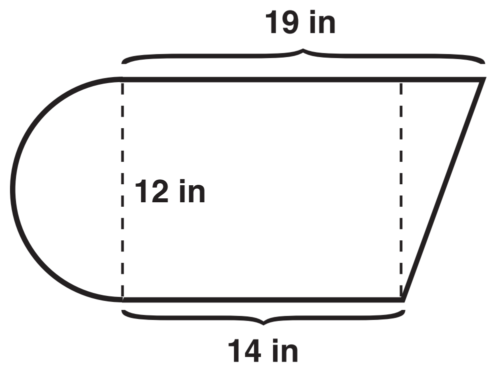
PERIMETER = ____________________

• Perimeter of Composite Figures •

Find the perimeter of each figure. Leave answers in exact form.

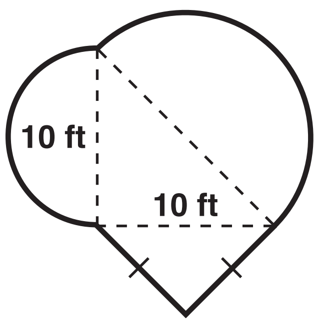


1)



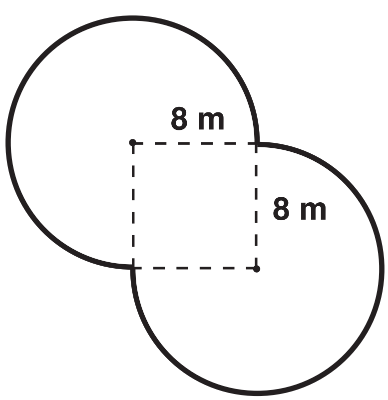
PERIMETER = ______________________________

2)



PERIMETER = ______________________________

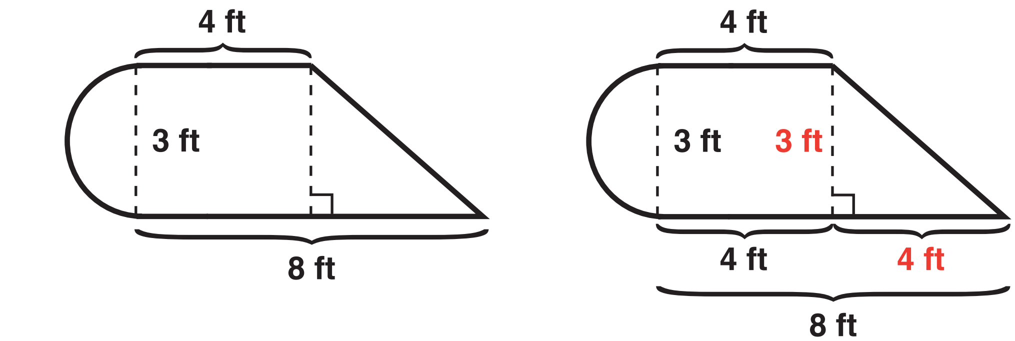
3)



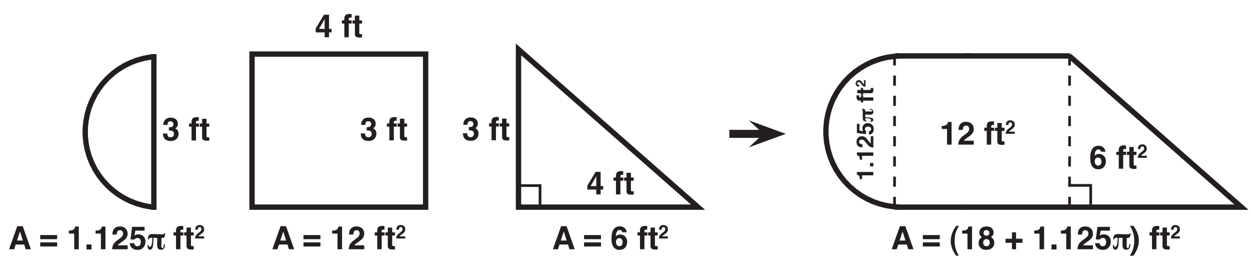
PERIMETER = ______________________________

• Area of Composite Figures •

Example: Find the area of the following figure.



To find the area of the whole composite figure, we add together the areas of each part of the figure. We can imagine the figure broken up as a semicircle, a rectangle, and a right triangle.



The area of the semicircle: $\frac{1}{2} \bullet \pi \bullet r^2 = \frac{1}{2} \bullet \pi \bullet (1.5)^2 = 1.125\pi \text{ ft}^2$

The area of the rectangle: $l \bullet w = 4 \bullet 3 = 12 \text{ ft}^2$

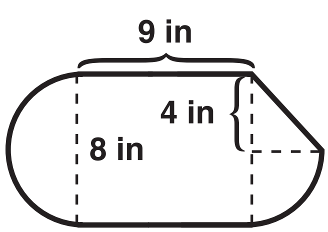
The area of the triangle: $\frac{1}{2} \bullet b \bullet h = \frac{1}{2} \bullet 4 \bullet 3 = 6 \text{ ft}^2$

So, the area of the whole figure is $12 + 6 + 1.125\pi = (18 + 1.125\pi) \text{ ft}^2$.

You may assume all angles that look right are right angles.

Try this: **Find the area of the following figure. Leave answer in exact form.**

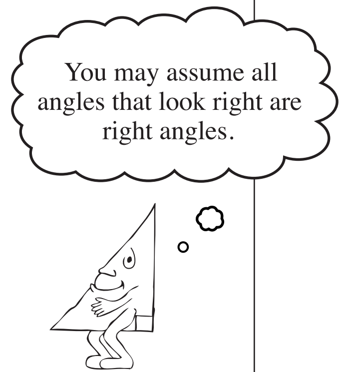
1)



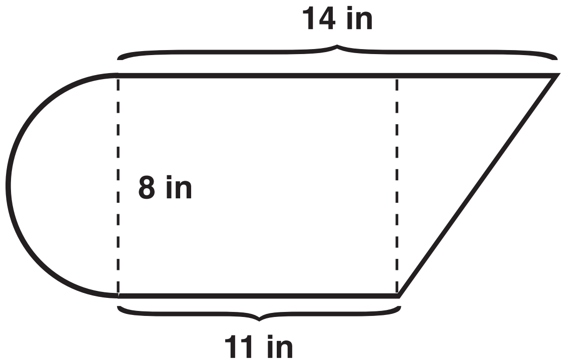
AREA = ______________________

• Area of Composite Figures •

Find the area of each figure. Leave answers in exact form.

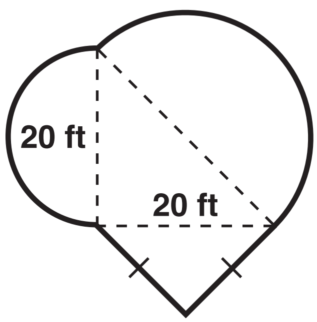


1)



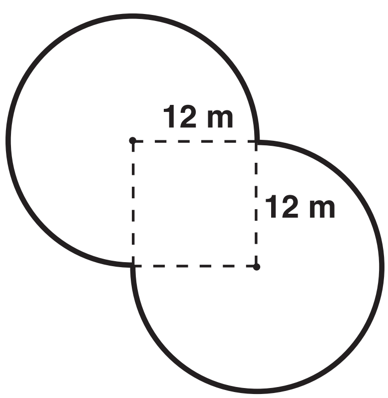
AREA = ______________________

2)

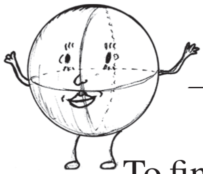


AREA = ______________________

3)



AREA = ______________________



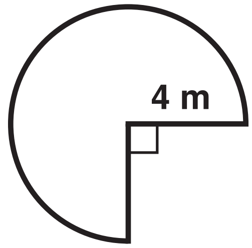
• Area of a Fraction of Circle •

Direct Teaching

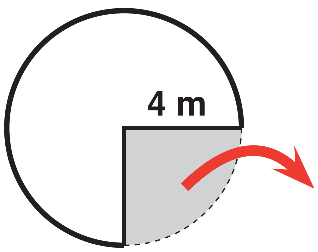
Wholes and Parts

To find the area of a composite figure, we can also subtract a part from a larger whole.

Example: Find the area of the following figure.



Here we have a three–quarter circle with a radius of 4 m. We can imagine this composite figure as a circle with a quarter circle missing from it.



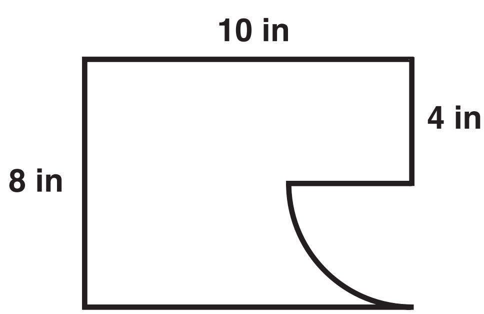
The area of the whole circle is $\pi(4)^2 = 16\pi$ m^2.

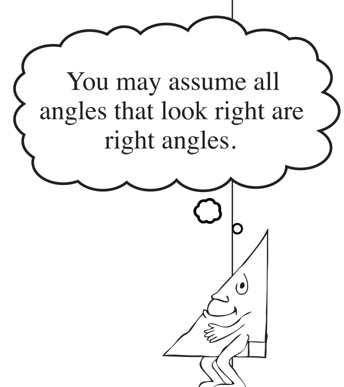
So, the area of the composite figure is the area of the whole circle (16π m^2) minus the area of the quarter circle ($\frac{1}{4} \bullet 16\pi$).

$$\Rightarrow 16\pi - (\tfrac{1}{4})16\pi = (\tfrac{3}{4})16\pi = 12\pi \text{ m}^2.$$

***Try this*: Find the area of the following figure. Leave answer in exact form.**

1)



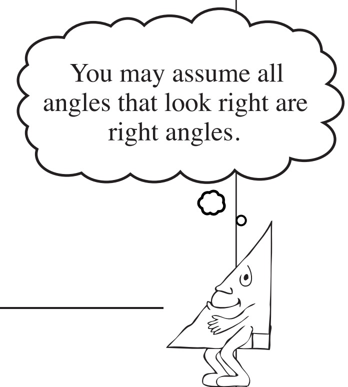


__________ – __________ = __________

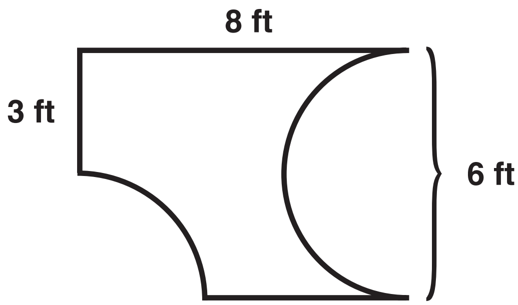
AREA OF RECTANGLE	AREA OF QUARTER CIRCLE	AREA OF COMPOSITE FIGURE

• Perimeter and Area of Composite Figures •

Find the perimeter and area of each figure. Leave answers in exact form.



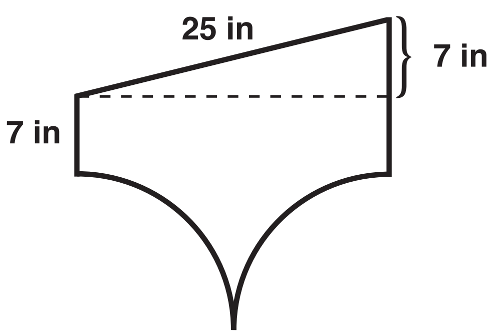
1)



PERIMETER = ____________________

AREA = ____________________

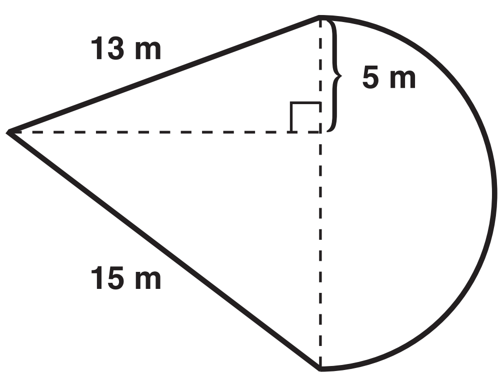
2)



PERIMETER = ____________________

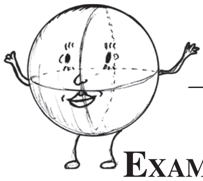
AREA = ____________________

3)



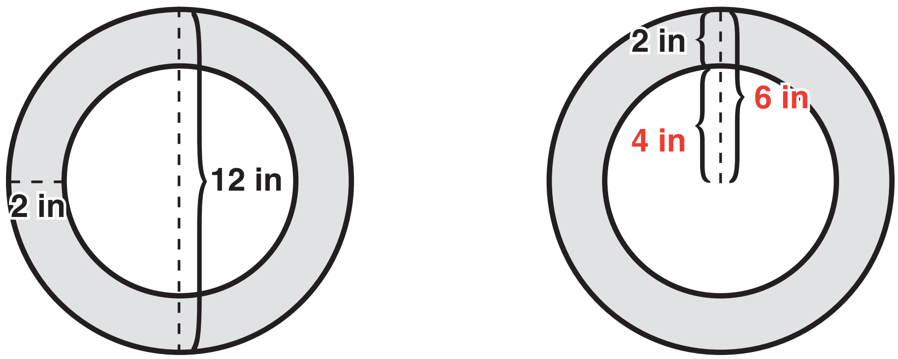
PERIMETER = ____________________

AREA = ____________________



• Area of Composite Figures • ☆

EXAMPLE: Find the area of the shaded region.



Since the diameter of the larger circle is 12 in, we know that its radius is 6 in. We can find the radius of the smaller circle (4 in) by subtracting the width of the shaded region (2 in) from the radius of the larger circle (6 in).

Here, the shaded region is a part of the whole larger circle. So, to find the area of the shaded region, we can subtract the area of the smaller circle from the area of the larger circle.

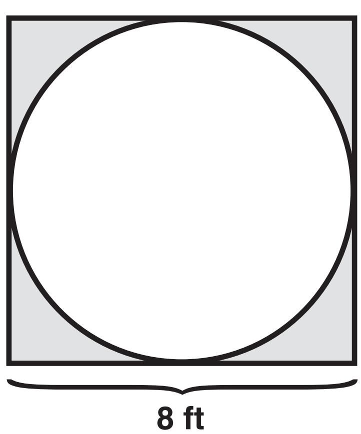
The area of the larger circle: $\pi \bullet r^2 = \pi \bullet 6^2 = 36\pi$ in^2

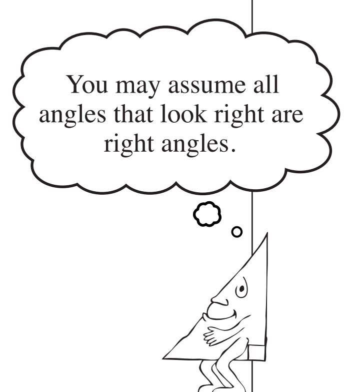
The area of the smaller circle: $\pi \bullet r^2 = \pi \bullet 4^2 = 16\pi$ in^2

So, the area of the shaded region is $36\pi - 16\pi = 20\pi$ in^2.

Try this: **Find the area of the shaded region. Leave answer in exact form.**

1)

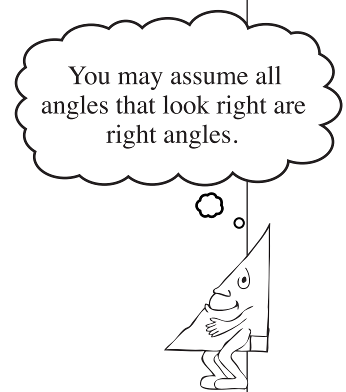




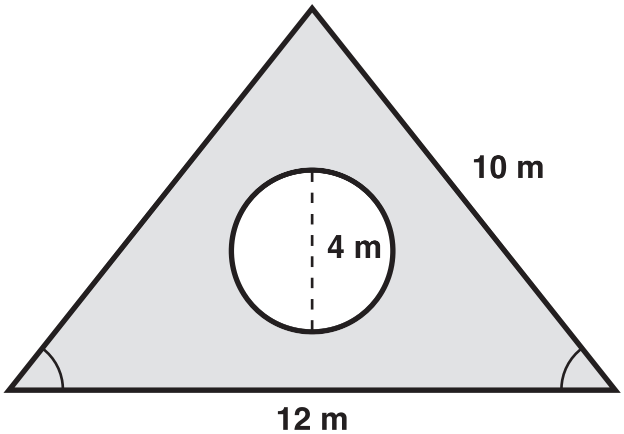
AREA = ____________________

• Area of Composite Figures •

Find the areas of the shaded regions. Leave answers in exact form.

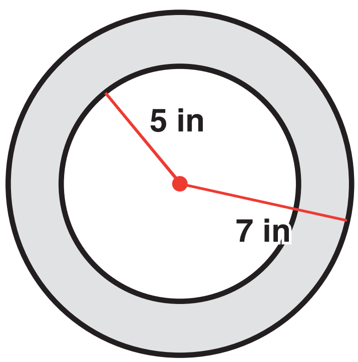


1)



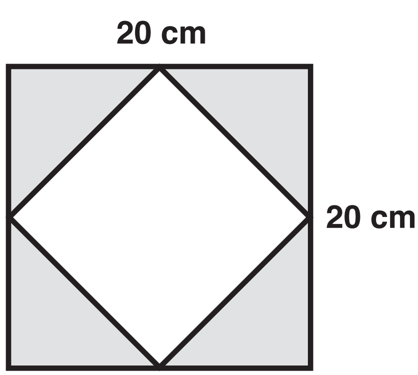
AREA = ____________________

2)

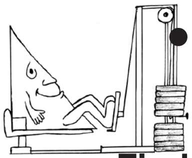


AREA = ____________________

3)



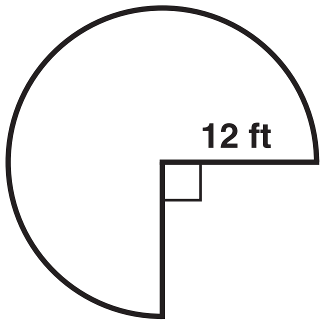
AREA = ____________________



• Mastery Check: Composite Figures •

Find the perimeter and area of the figure. Leave answers in exact form.

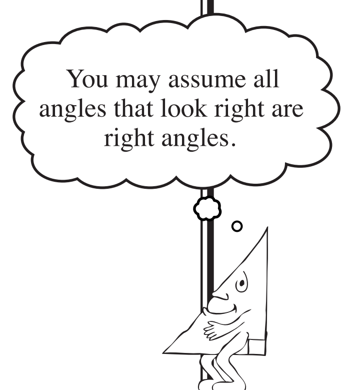
1)



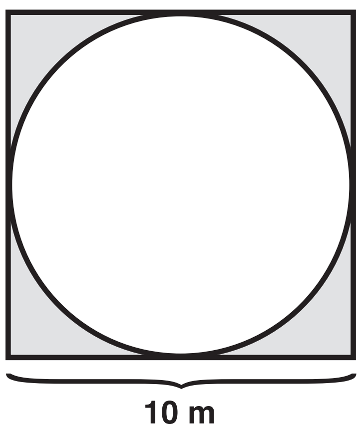
PERIMETER = ______________________________

AREA = ______________________________

Find the area of the shaded region. Leave answer in exact form.



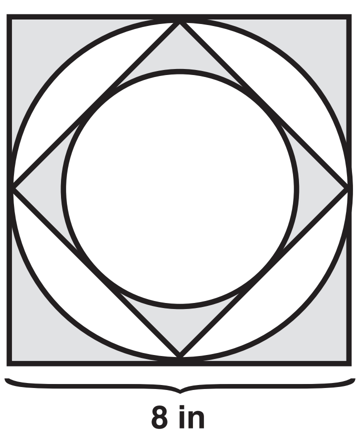
2)



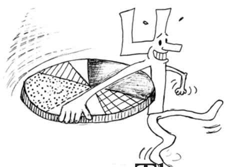
AREA = ______________________________

Challenge: **Find the area of the shaded region. Leave answer in exact form.**

3)



AREA = ______________________________



• 45° – 45° – 90° Triangles •

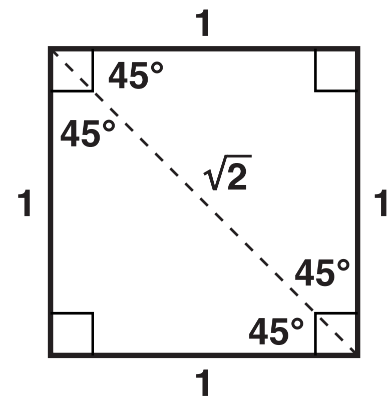
Direct Teaching

There are two right triangles that we refer to as **SPECIAL RIGHT TRIANGLES**. They are right triangles that have simple relationships between side lengths, making the triangles' measurements useful in certain calculations and formulas.

One of the **SPECIAL RIGHT TRIANGLES** is called a **45° – 45° – 90° TRIANGLE**. This triangle has two 45° angles and one 90° angle. It is an **ISOSCELES TRIANGLE** because two of its sides are the same length.

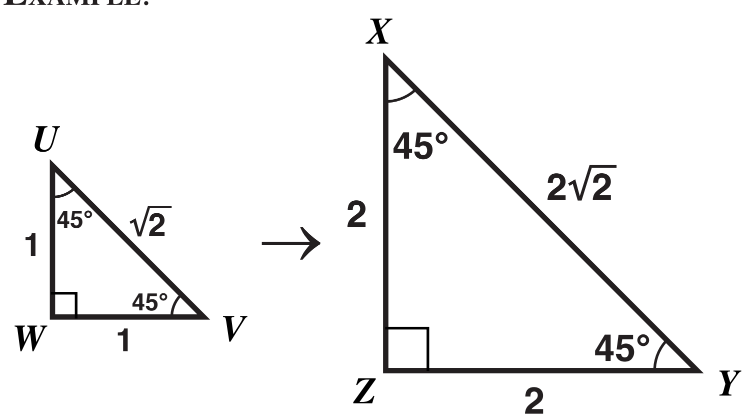
The square to the right has side lengths of 1. Splitting it in half diagonally will result in two **45° – 45° – 90° TRIANGLES**. Using the Pythagorean Theorem, we can calculate the length of the hypotenuse.

$$\boldsymbol{a}^2 + \boldsymbol{b}^2 = \boldsymbol{c}^2$$
$$1^2 + 1^2 = 2 = \boldsymbol{c}^2$$
$$\boldsymbol{c} = \sqrt{2}$$



The side lengths of a **45° – 45° – 90° TRIANGLE** are not always equal to 1, 1, and $\sqrt{2}$, but this relationship between the side lengths can be generalized to any **45° – 45° – 90° TRIANGLE** because triangles with congruent angles are similar.

EXAMPLE:



$\boldsymbol{UW} : \boldsymbol{XZ} = 1 : 2$

$\boldsymbol{VW} : \boldsymbol{YZ} = 1 : 2$

$\boldsymbol{UV} : \boldsymbol{XY} = \sqrt{2} : 2\sqrt{2} = 1 : 2$

The scale factor of $\triangle UVW$ to $\triangle XYZ$ is 1 : 2. We can multiply each side of $\triangle UVW$ by 2 to get those of $\triangle XYZ$.

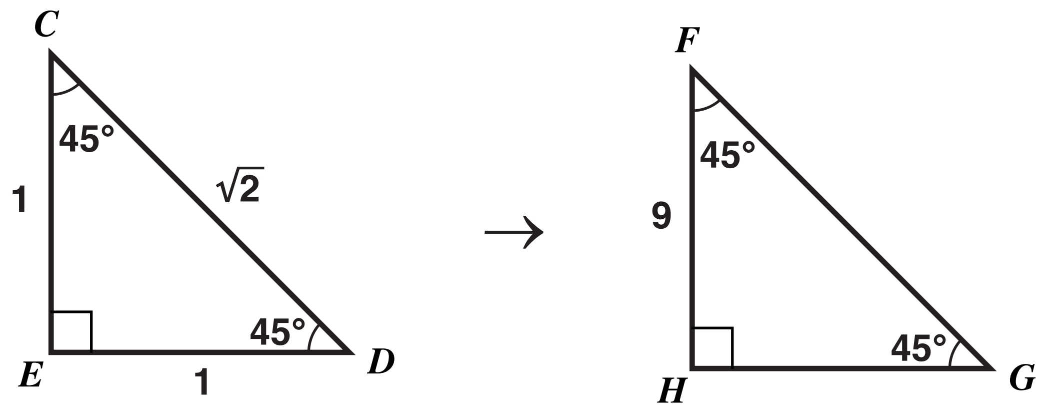
Try this: **Find the scale factor using $\triangle UVW$ in the example above for comparison.**

1) A 45° – 45° – 90° triangle has the side lengths 7, 7, and $7\sqrt{2}$.

SCALE FACTOR: _______ : _______

• 45° – 45° – 90° Triangles •

Example: Find the scale factor and the missing side lengths of $\triangle FGH$.



The ratio of CE to FH is 1 : 9. This means that the scale factor of $\triangle CDE$ to $\triangle FGH$ is 1 : 9. We can multiply each side of $\triangle CDE$ by 9 to get the sides of $\triangle FGH$.

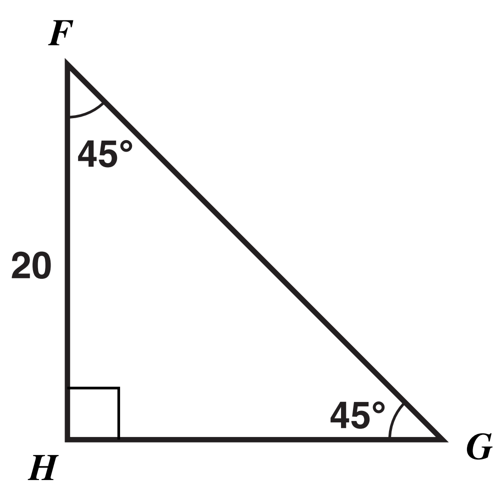
Since $\triangle FGH$ is an isosceles triangle, $\overline{FH} \cong \overline{GH}$. So, GH is also equal to 9. We can use the scale factor to find FG.

$$FG = CD \bullet 9$$

$$FG = \sqrt{2} \bullet 9 = 9\sqrt{2}$$

Try this: **Find the scale factor and the missing side lengths. Use $\triangle CDE$ in the example above for comparison.**

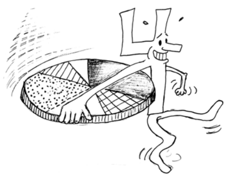
1)



SCALE FACTOR: 1 : ______

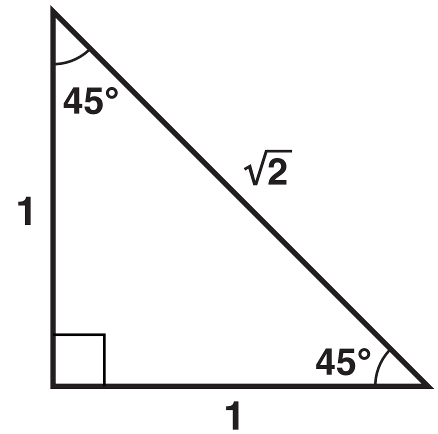
GH = ____________

FG = ____________



• 45° – 45° – 90° Triangles •

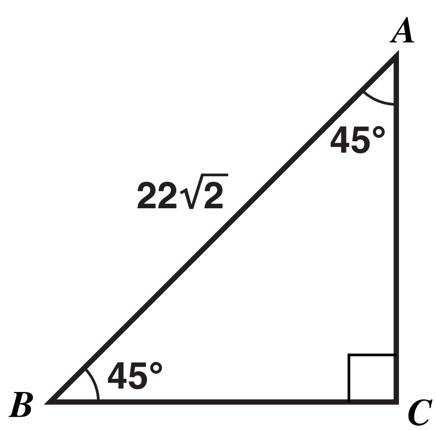
45° – 45° – 90° Triangle



The legs are the same and the hypotenuse is $\sqrt{2}$ times the leg.

Find the scale factor using the triangle above for comparison and the missing side lengths.

1)

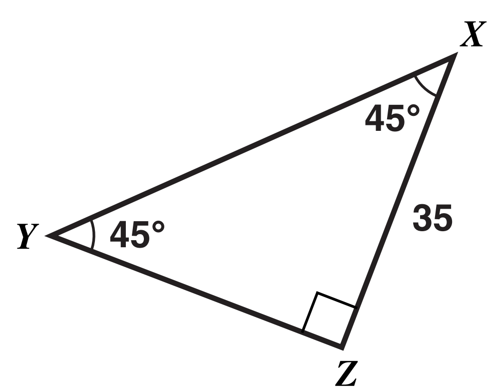


SCALE FACTOR: ________ : ________

AC = ______________

BC = ______________

2)



SCALE FACTOR: ________ : ________

XY = ______________

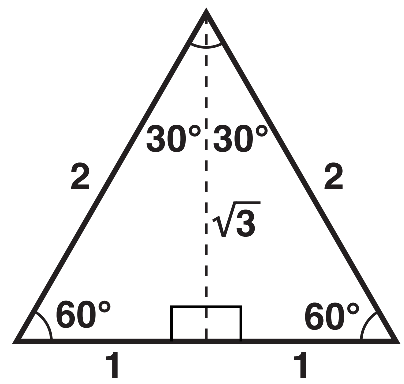
YZ = ______________

• 30° – 60° – 90° Triangles •

Direct Teaching

The other **special right triangle** is called a **30° – 60° – 90° triangle**. This triangle has a 30° angle, a 60° angle, and a 90° angle.

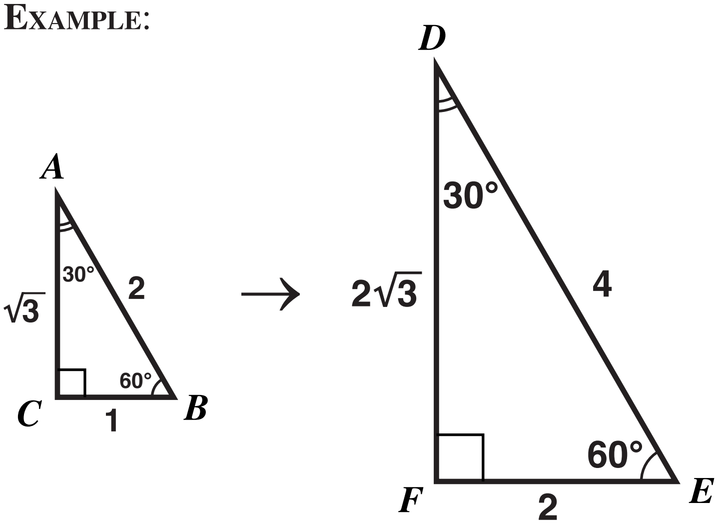
The equilateral triangle to the right has side lengths of 2. Splitting it in half will result in two **30° – 60° – 90° triangles**. The shorter leg is the result of bisecting one of the sides of the original triangle. Using the Pythagorean Theorem, we can calculate the length of the longer leg.



$$a^2 + b^2 = c^2$$
$$1^2 + b^2 = 2^2$$
$$b^2 = 3$$
$$b = \sqrt{3}$$

The side lengths of a **30° – 60° – 90° triangle** are not always equal to 1, $\sqrt{3}$, and 2, but this relationship between the side lengths can be generalized to any **30° – 60° – 90° triangle** because triangles with congruent angles are similar.

Example:



$AB : DE = 2 : 4 = 1 : 2$

$BC : EF = 1 : 2$

$AC : DF = \sqrt{3} : 2\sqrt{3} = 1 : 2$

The scale factor of $\triangle ABC$ to $\triangle DEF$ is 1 : 2. We can multiply each side of $\triangle ABC$ by 2 to get those of $\triangle DEF$.

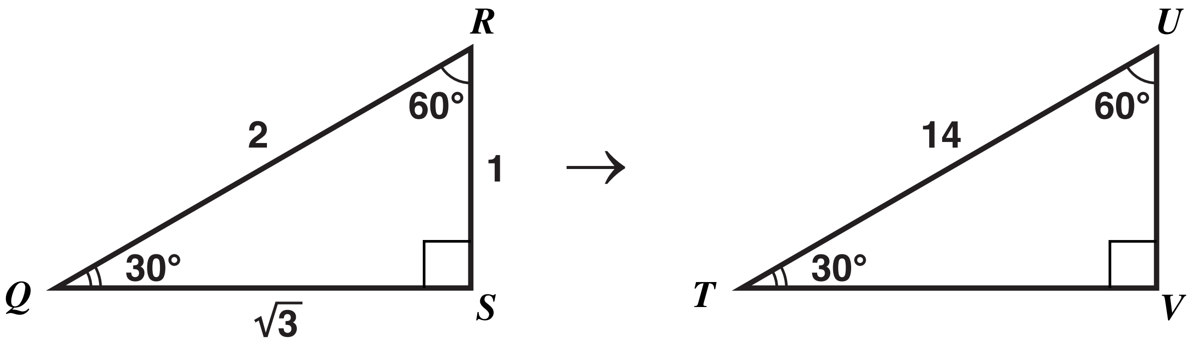
Try this:

1) In a 30° – 60° – 90° triangle, the shorter leg is opposite the ______° angle and the longer leg is opposite the ______° angle.

• 30° – 60° – 90° Triangles •

Direct Teaching

EXAMPLE: Find the scale factor and the missing side lengths of $\triangle TUV$.



The ratio of QR to TU is 1 : 7. This means that the scale factor of $\triangle QRS$ to $\triangle TUV$ is 1 : 7. We can multiply each side of $\triangle QRS$ by 7 to get the sides of $\triangle TUV$.

We can use the scale factor to find UV and TV.

$UV = RS \cdot 7$

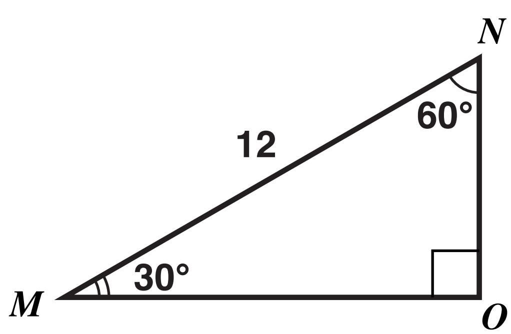
$UV = 1 \cdot 7 = 7$

$TV = QS \cdot 7$

$TV = \sqrt{3} \cdot 7 = 7\sqrt{3}$

Try this: **Find the scale factor and the missing side lengths. Use $\triangle QRS$ in the example above for comparison.**

1)



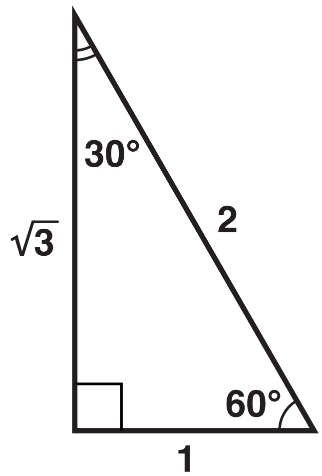
SCALE FACTOR: _______ : _______

MO = _____________

NO = _____________

• 30° – 60° – 90° Triangles •

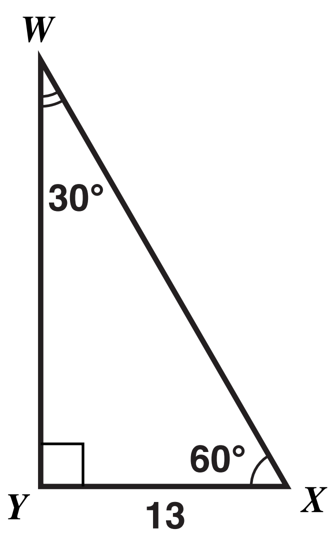
30° – 60° – 90° Triangle



The hypotenuse is twice the shorter leg and the longer leg is $\sqrt{3}$ times the shorter leg.

Find the scale factor using the triangle above for comparison and the missing side lengths.

1)

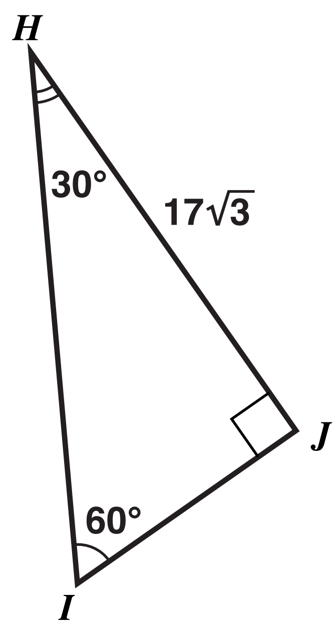


SCALE FACTOR: ________ : ________

WX = ________________

WY = ________________

2)



SCALE FACTOR: ________ : ________

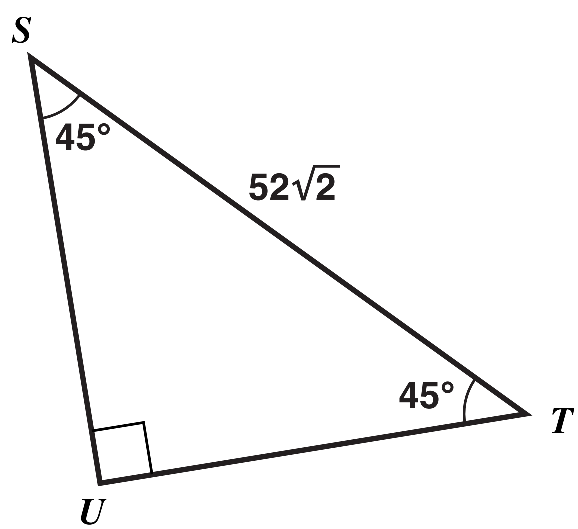
IJ = ________________

HI = ________________

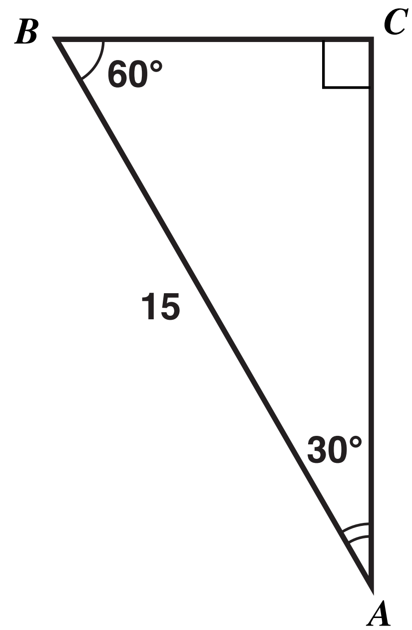
• Special Right Triangles •

Find the missing side lengths.

1)



2)



SU = ____________

AC = ____________

TU = ____________

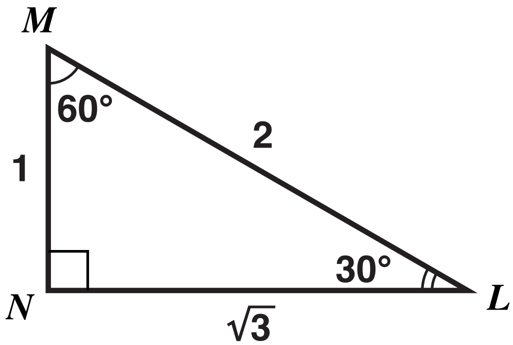
BC = ____________

3) A triangle with side lengths 26, $13\sqrt{3}$, and 13 is a ______________ triangle.

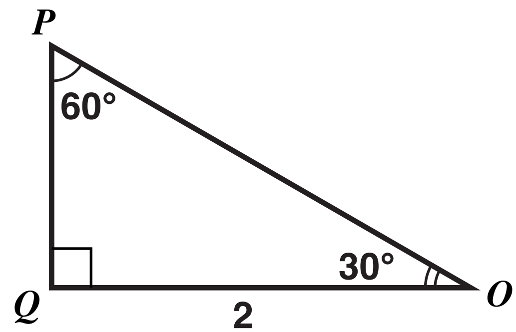
4) A triangle with side lengths $63\sqrt{2}$, 63, and 63 is a ______________ triangle.

• Special Right Triangles •

Example: Find the scale factor and the missing side lengths.



→



The ratio of ***NL*** to ***QO*** is $\sqrt{3} : 2$. Let's simplify this ratio.

$$\sqrt{3} : 2$$

$$\frac{\sqrt{3}}{\sqrt{3}} : \frac{2}{\sqrt{3}} = 1 : \frac{2}{\sqrt{3}}$$

Since ***QO*** is larger than ***NL***, $\triangle$***OPQ*** is larger than $\triangle$***LMN*** by $\frac{2}{\sqrt{3}}$. So, we can find the missing sides lengths of $\triangle$***OPQ*** by multiplying the side lengths of $\triangle$***LMN*** by $\frac{2}{\sqrt{3}}$.

$$PQ = MN \cdot \frac{2}{\sqrt{3}} \qquad OP = LM \cdot \frac{2}{\sqrt{3}}$$

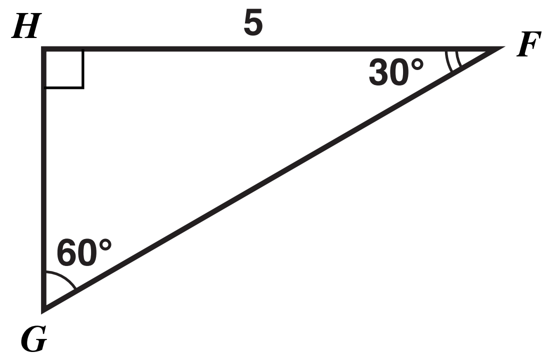
$$PQ = 1 \cdot \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \qquad OP = 2 \cdot \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

Now, we must rationalize the side lengths.

$$PQ = \frac{2}{\sqrt{3}} = \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad OP = \frac{4}{\sqrt{3}} = \frac{4 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{4\sqrt{3}}{3}$$

Try this: **Find the scale factor using $\triangle$*LMN* in the example above for comparison and the missing side lengths.**

1)



SCALE FACTOR: _______ : _______

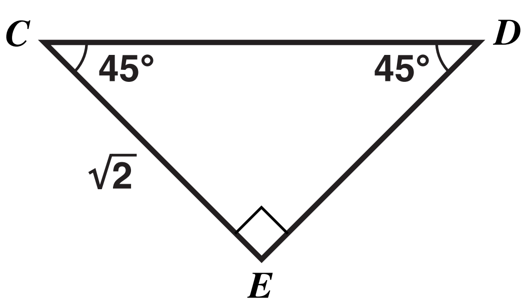
GH = ______________

FG = ______________

• Special Right Triangles •

Find the missing side lengths.

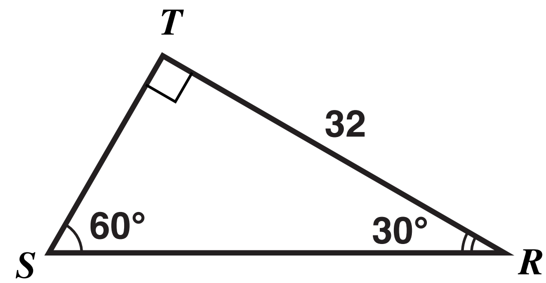
1)



DE = ______________

CD = ______________

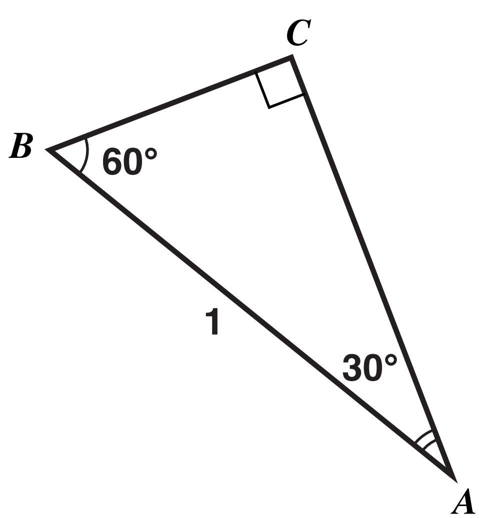
2)



ST = ______________

RS = ______________

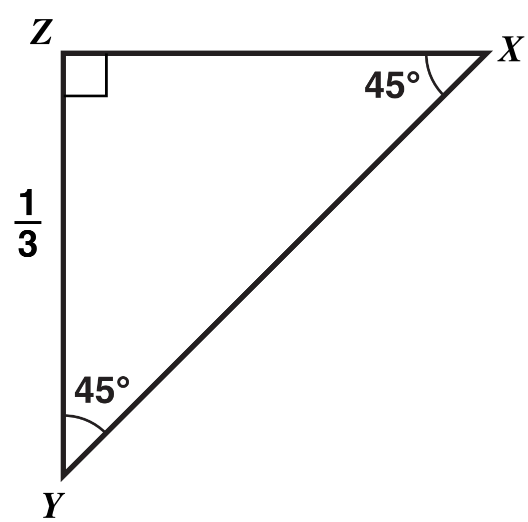
3)



BC = ______________

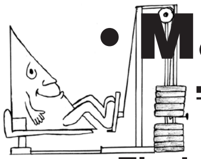
AC = ______________

4)



XZ = ______________

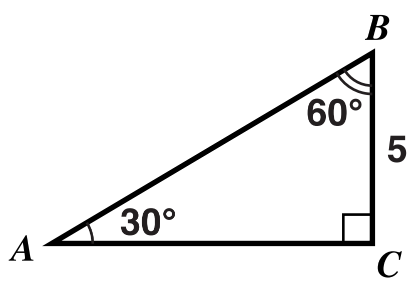
XY = ______________



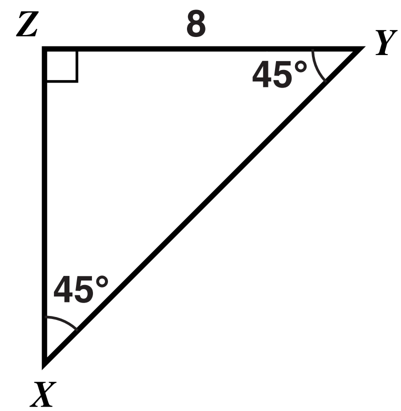
• Mastery Check: Special Right Triangles • ☆

Find the missing side lengths.

1)



2)



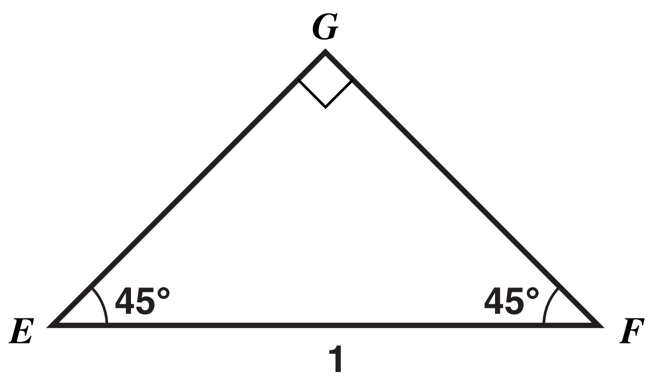
AB = ______________

AC = ______________

XZ = ______________

XY = ______________

3)

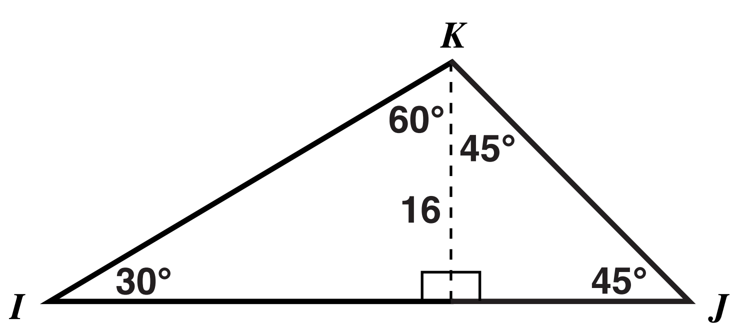


EG = ______________

FG = ______________

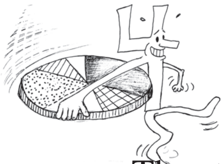
Challenge:

4)



PERIMETER = ________________

AREA = ________________



• Trigonometric Ratios: Sine •

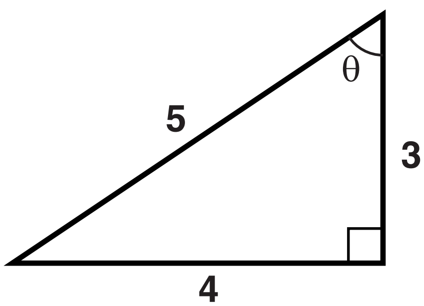
Direct Teaching

The study of the relationships between the angles and the sides of triangles is called **TRIGONOMETRY**. The ratios of certain side lengths of any right triangle are called **TRIGONOMETRIC RATIOS**. Let's take a look at one of the most common ratios, **SINE**.

The **SINE** of an acute angle in a right triangle is equal to the ratio of the angle's **OPPOSITE** side and the triangle's **HYPOTENUSE**. It is commonly abbreviated as "**SIN**."

$$\text{SIN}(\theta) = \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}}$$

EXAMPLE: Find SIN(θ).

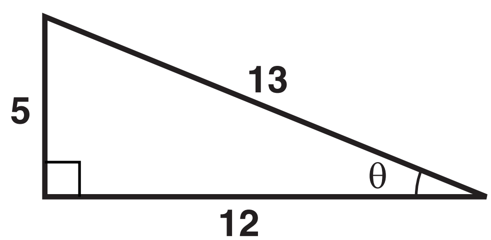


In this case, the **OPPOSITE** side to angle θ is equal to **4** and the **HYPOTENUSE** of the right triangle is equal to **5**.

Since the **SINE** of an angle in a right triangle is equal to its **OPPOSITE** side over the **HYPOTENUSE**, the **SINE** of θ is:

$$\text{SIN}(\theta) = \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}} = \frac{4}{5}$$

Try these: **Use the triangle below to solve the following problems.**



1) The opposite side to θ is equal to ________.

2) The hypotenuse is equal to ________.

3) SIN(θ) = ________

4) Can SIN(θ) ever be greater than or equal to 1? Explain. ________________

____

• Trigonometric Ratios: Cosine •

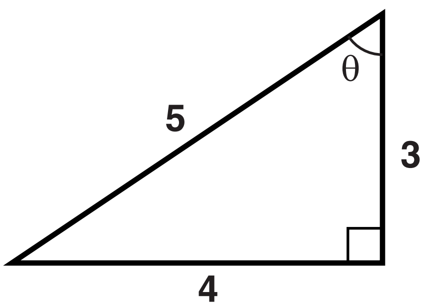
Direct Teaching

Let's take a look at the ratio **COSINE**.

The **COSINE** of an acute angle in a right triangle is equal to the ratio of the angle's **ADJACENT** side and the triangle's **HYPOTENUSE**. It is commonly abbreviated as "**COS**."

$$\cos(\theta) = \frac{\text{ADJACENT}}{\text{HYPOTENUSE}}$$

EXAMPLE: Find $\cos(\theta)$.

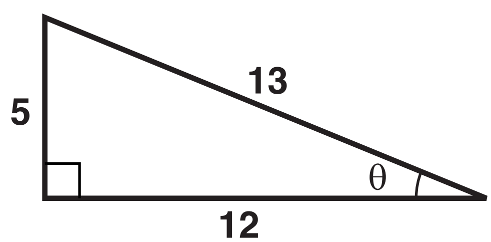


In this case, the **ADJACENT** side to angle θ is equal to **3** and the **HYPOTENUSE** of the right triangle is equal to **5**.

Since the **COSINE** of an angle in a right triangle is equal to its **ADJACENT** side over the **HYPOTENUSE**, the **COSINE** of θ is:

$$\cos(\theta) = \frac{\text{ADJACENT}}{\text{HYPOTENUSE}} = \frac{3}{5}$$

Try these: **Use the triangle below to solve the following problems.**



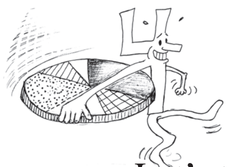
1) The adjacent side to θ is equal to ________.

2) The hypotenuse is equal to ________.

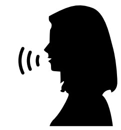
3) $\cos(\theta)$ = ________

4) Can $\cos(\theta)$ ever be greater than or equal to 1? Explain. ____________________

____



• Trigonometric Ratios: Tangent •



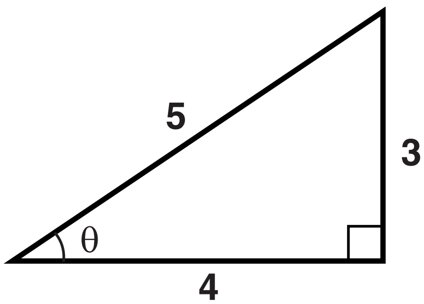
Direct Teaching

Let's take a look at the ratio TANGENT.

The TANGENT of an acute angle in a right triangle is equal to the ratio of the angle's OPPOSITE side and its ADJACENT side. It is commonly abbreviated as "TAN."

$$\text{TAN}(\theta) = \frac{\text{OPPOSITE}}{\text{ADJACENT}}$$

EXAMPLE: Find TAN(θ).

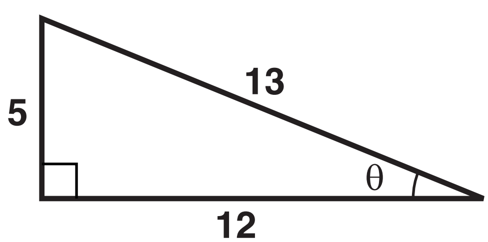


In this case, the OPPOSITE side to angle θ is equal to **3** and the ADJACENT side to angle θ is equal to **4**.

Since the TANGENT of an angle in a right triangle is equal to its OPPOSITE side over its ADJACENT side, the TANGENT of θ is:

$$\text{TAN}(\theta) = \frac{\text{OPPOSITE}}{\text{ADJACENT}} = \frac{3}{4}$$

Try these: **Use the triangle below to solve the following problems.**



1) The opposite side to θ is equal to ________.

2) The adjacent side to θ is equal to ________. **3**) TAN(θ) = ________

4) Can TAN(θ) ever be greater than or equal to 1? Explain. ____________________

____

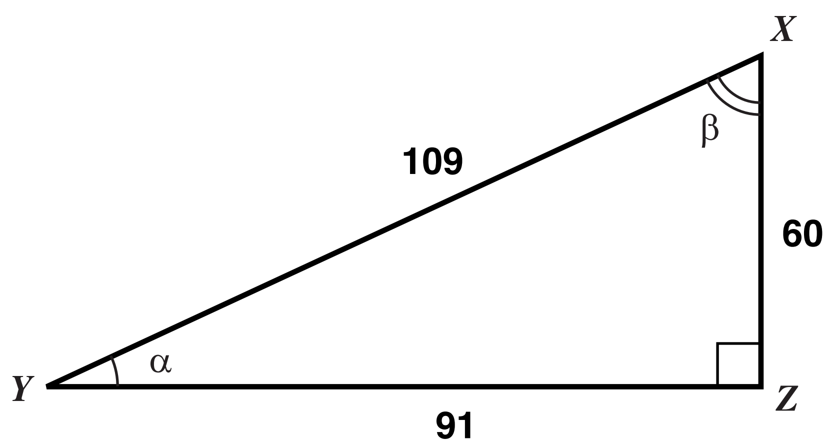
• Trigonometric Ratios •

$$\text{SIN}(\theta) = \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}} = \frac{\text{O}}{\text{H}}$$

$$\text{COS}(\theta) = \frac{\text{ADJACENT}}{\text{HYPOTENUSE}} = \frac{\text{A}}{\text{H}}$$

$$\text{TAN}(\theta) = \frac{\text{OPPOSITE}}{\text{ADJACENT}} = \frac{\text{O}}{\text{A}}$$

SOH-CAH-TOA



Use Δ*XYZ* to solve the following problems.

1) $\text{SIN}(\alpha) =$ __________

2) $\text{COS}(\alpha) =$ __________

3) $\text{TAN}(\alpha) =$ __________

4) $\text{SIN}(\beta) =$ __________

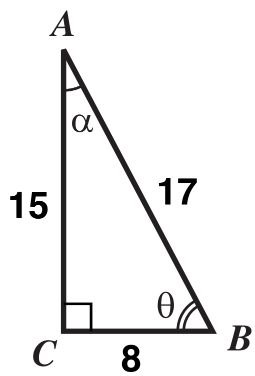
5) $\text{COS}(\beta) =$ __________

6) $\text{TAN}(\beta) =$ __________

• Trigonometric Ratios •

All TRIGONOMETRIC RATIOS should be written as reduced fractions. Do not convert any improper fractions to mixed numbers.

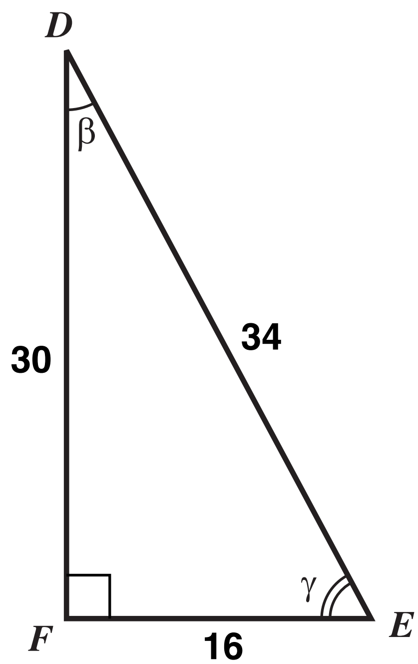
EXAMPLE: $\triangle ABC$ and $\triangle DEF$ are similar triangles.



$\sin(\alpha) = \mathbf{\frac{8}{17}}$

$\cos(\alpha) = \mathbf{\frac{15}{17}}$

$\tan(\alpha) = \mathbf{\frac{8}{15}}$



$\sin(\beta) = \frac{16}{34} = \mathbf{\frac{8}{17}}$

$\cos(\beta) = \frac{30}{34} = \mathbf{\frac{15}{17}}$

$\tan(\beta) = \frac{16}{30} = \mathbf{\frac{8}{15}}$

Since $\triangle ABC$ and $\triangle DEF$ are similar, the corresponding angles are congruent. The TRIGONOMETRIC RATIOS of congruent angles are equal.

***Try these*: Use the two triangles above. Reduce all ratios.**

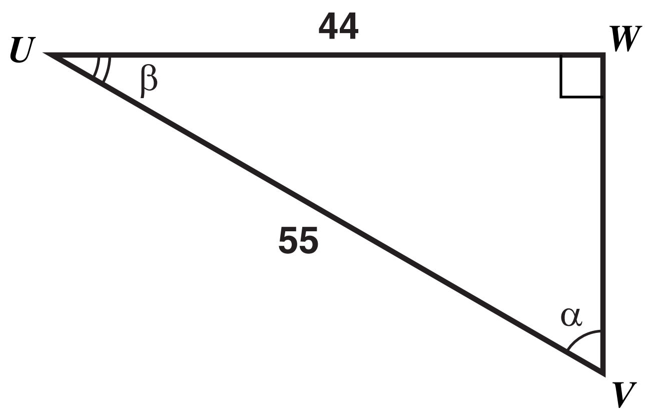
1) **a)** $\sin(\theta)$ = ________ **b)** $\cos(\theta)$ = ________ **c)** $\tan(\theta)$ = ________

2) Without making any calculations, find the following:

a) $\sin(\gamma)$ = ________ **b)** $\cos(\gamma)$ = ________ **c)** $\tan(\gamma)$ = ________

• Trigonometric Ratios •

Use △*UVW* to solve the following problems. Reduce all fractions.



Use a Calculator

1) Using the Pythagorean Theorem, find and label the missing side length.

2) SIN(α) = ________

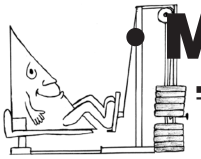
3) COS(α) = ________

4) TAN(α) = ________

5) SIN(β) = ________

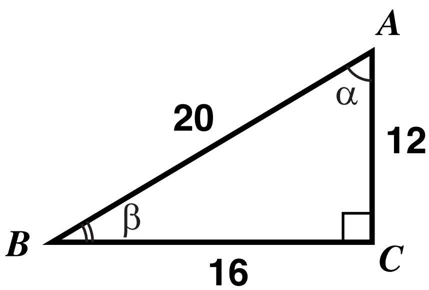
6) COS(β) = ________

7) TAN(β) = ________



• Mastery Check: Trigonometric Ratios •

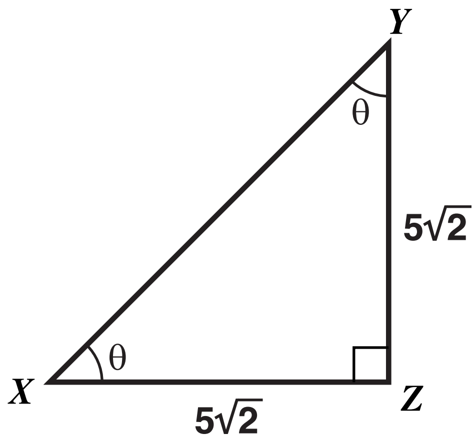
Reduce all fractions.



1) SIN(α) = ________

2) COS(α) = ________

3) TAN(β) = ________

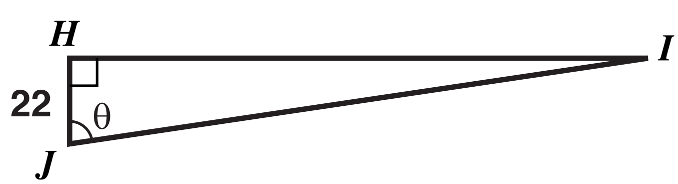


4) Using the Pythagorean Theorem, find and label the missing side length.

5) SIN(θ) = ________ 6) COS(θ) = ________ 7) TAN(θ) = ________

Challenge:

8)



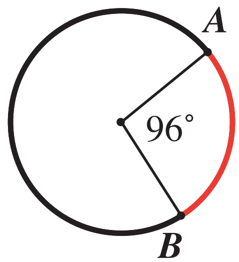
If TAN$(\theta) = \frac{60}{11}$, what is the length of $\overline{HI}$?

HI = ________

• Arc Measure •

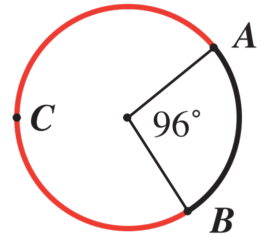
Direct Teaching

Wholes and Parts



The measure of the MINOR ARC of a circle is equal to the measure of its central angle.

$m\overset{\frown}{AB} = 96^\circ$

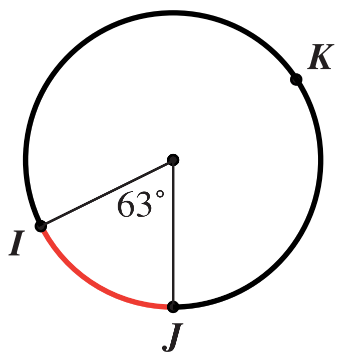


The measure of the MAJOR ARC of a circle is equal to the degrees around the circle (the whole = 360°) minus the measure of the central angle (the part = 96°).

$m\overset{\frown}{ACB} = 360^\circ - 96^\circ = 264^\circ$

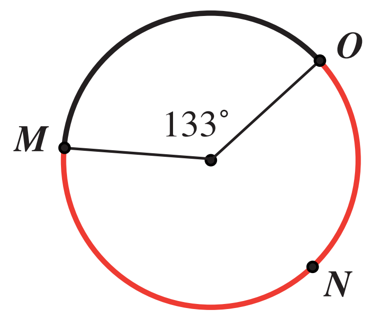
Try these:

1)



$m\overset{\frown}{IJ} =$ ____________

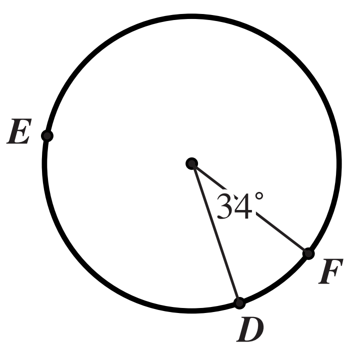
2)



$360^\circ - m\overset{\frown}{MO} =$ ____________

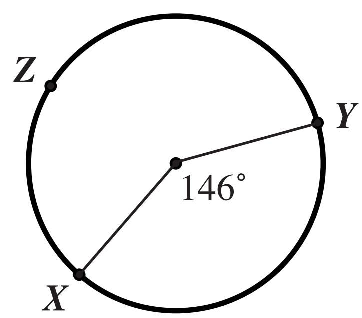
$m\overset{\frown}{MNO} =$ ____________

3)



$m\overset{\frown}{DF} =$ ____________

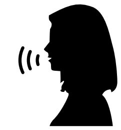
4)



$360^\circ - m\overset{\frown}{XY} =$ ____________

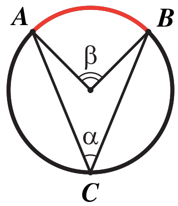
$m\overset{\frown}{XZY} =$ ____________

• Arc Measures •



Direct Teaching

Inscribed Angle Theorem



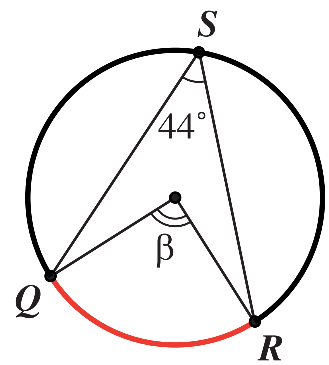
The measure of an inscribed angle is equal to half the measure of the central angle that is subtended by the same arc.

$\beta = 2\alpha$

The measure of an arc is equal to twice the measure of the inscribed angle that it SUBTENDS.

EXAMPLE: Find the measure of the minor arc $\overset{\frown}{QR}$.

We can find the measure of the central angle of $\overset{\frown}{QR}$ using the Inscribed Angle Theorem.

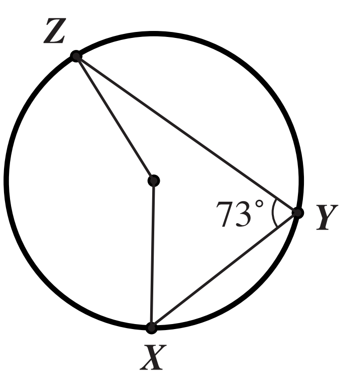


$\beta = 2\alpha$
$\beta = 2 \bullet 44^\circ = 88^\circ$

The measure of a minor arc is equal to the measure of its central angle. So, $m\overset{\frown}{QR} = 88^\circ$.

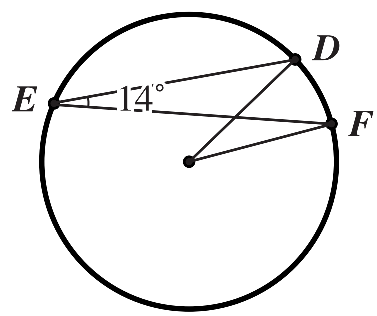
Try these: **Find the measure of the minor arc.**

1)



$m\overset{\frown}{XZ} =$ ____________

2)

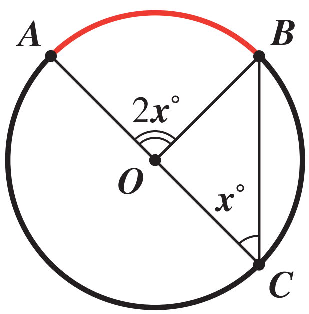
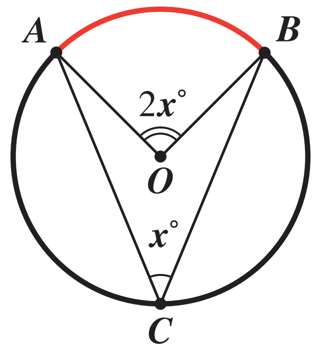
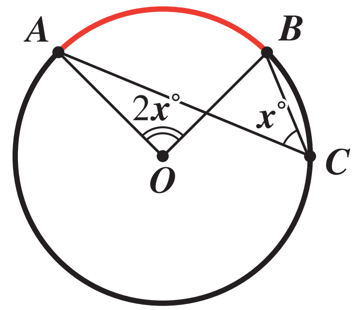


$m\overset{\frown}{DF} =$ ____________

• Inscribed Angles: Three Cases •

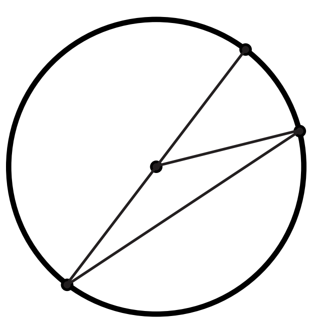
There are three unique ways that an INSCRIBED ANGLE relates to the central angle of the minor arc that SUBTENDS them both.

In the examples below, the central angle, $\angle AOB$, and the INSCRIBED ANGLE, $\angle ACB$, are SUBTENDED by the minor arc $\overset{\frown}{AB}$.

Case 1	Case 2	Case 3
		
One of the chords of the inscribed angle is the diameter.	The center of the circle is between the chords of the inscribed angle.	The center of the circle is outside the chords of the inscribed angle.

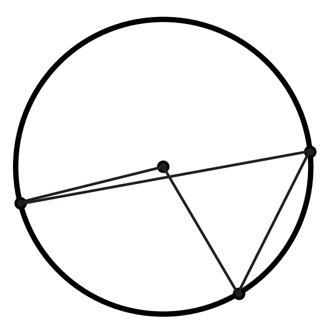
Try these: **Identify each inscribed angle as Case 1, Case 2, or Case 3.**

1)



CASE ______

2)

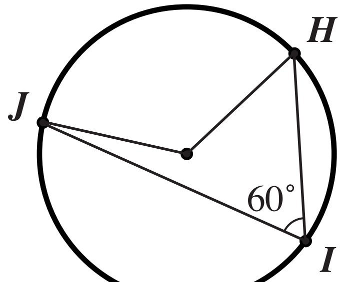


CASE ______

• Arc Measures •

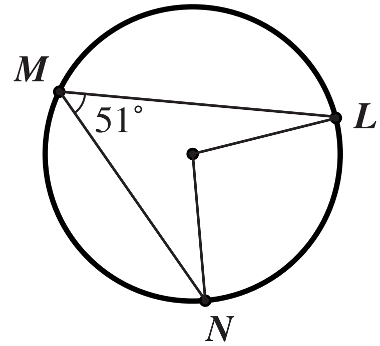
☆

1)



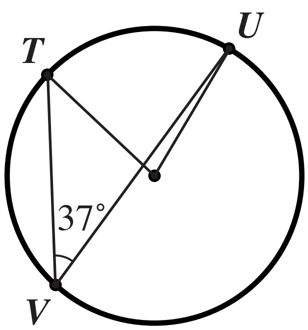
$m\overset{\frown}{HJ}$ = ____________

2)



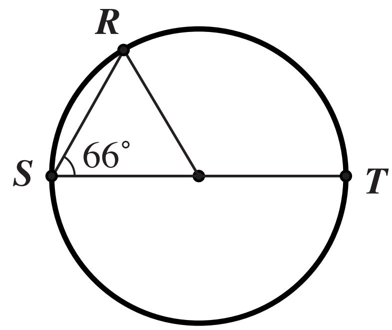
$m\overset{\frown}{LN}$ = ____________

3)



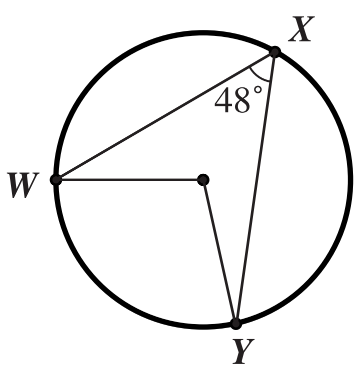
$m\overset{\frown}{TVU}$ = ____________

4)



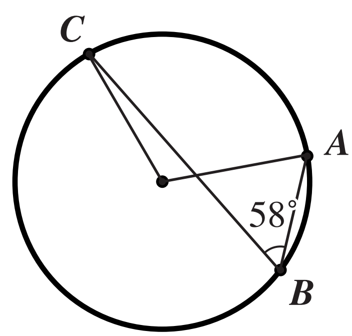
$m\overset{\frown}{RT}$ = ____________

5)



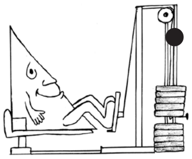
$m\overset{\frown}{WY}$ = ____________

6)



$m\overset{\frown}{ABC}$ = ____________

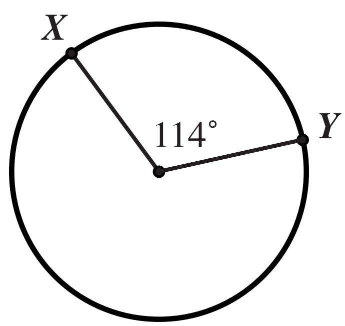
Proof Bypass



• **Mastery Check:** Measure of an Arc •

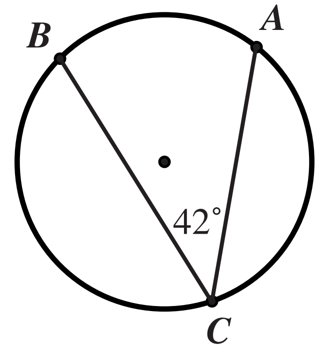
☆

1)



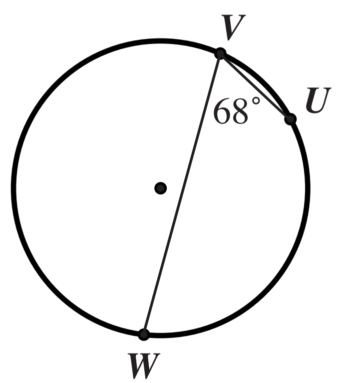
$m\overset{\frown}{XY} =$ ____________

2)



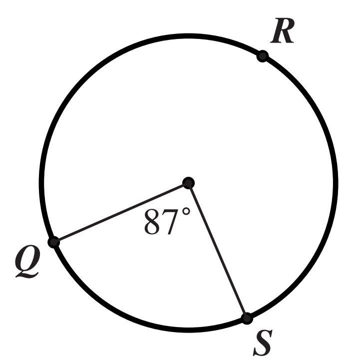
$m\overset{\frown}{AB} =$ ____________

3)



$m\overset{\frown}{UW} =$ ____________

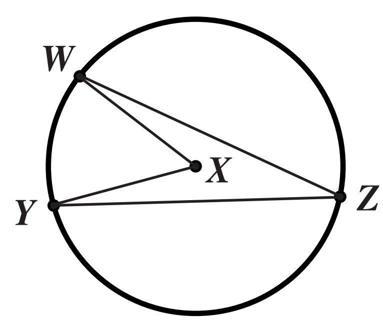
4)



$m\overset{\frown}{QRS} =$ ____________

Challenge:

5) Find $m\angle WZY$ when $\overset{\frown}{WZY} = 309^\circ$.



$m\angle WZY =$ ____________

• Inverses •

Direct Teaching

Most operations in math have an opposite. For example, the opposite of addition is subtraction. The opposite of multiplication is division. The opposite of squaring a number is taking the square root. Similarly, there is an opposite to applying a function; it is called an INVERSE FUNCTION. Inverse functions are important because they help *undo* any operation done by the original function. The symbol for an inverse function is $f^{-1}(x)$.

***Try this*: Fill in the blanks.**

Let's look at the function $f(x) = x + 1$. In this function, we take the input value and add 1 to get the output value. Let's list some values.

x	$f(x)$
0	1
1	
2	
3	
4	

Now, let's take a look at the following table. Let's find the pattern and fill in the table.

x	$f^{-1}(x)$
1	0
2	1
3	2
4	
5	

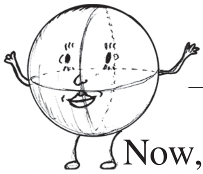
What was the rule you used?

____

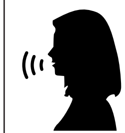
Notice how in the first table, we add 1 to get $f(x)$, but then in the second table, we subtracted 1 to get x. In fact, if you look at the two tables, you'll notice that the numbers in the columns are the same, just switched.

In other words, the second table represents the INVERSE FUNCTION of $f(x)$.

So, the inverse function of $f(x) = x + 1$ is $f^{-1}(x) =$ _______________ .



• Inverse Lines Using Points •



Direct Teaching

Now, let's look at the equation $\boldsymbol{y} = 2\boldsymbol{x} + 6$ and let's find two points on the line.

x	y	Point
1	8	(1, 8)
2	10	(2, 10)

So now we have two points for this line.

Let's switch the coordinates of the points: Now, we have the points (8, 1) and (10, 2).

Since we have two points, we can find the equation of the line.

The equation of the line that goes through (8, 1) and (10, 2) is $\boldsymbol{y} = \frac{1}{2}\boldsymbol{x} - 3$.

Now, let's compare more points between the two equations.

$y = 2x + 6$		
x	y	Point
3	12	(3, 12)
4	14	(4, 14)
5	16	(5, 16)

$y = \frac{1}{2}x - 3$		
x	y	Point
12	3	(12, 3)
14	4	(14, 4)
16	5	(16, 5)

The points are flipped between the two tables for each point: $(\boldsymbol{a}, \boldsymbol{b})$ for $\boldsymbol{y} = 2\boldsymbol{x} + 6$ is $(\boldsymbol{b}, \boldsymbol{a})$ for $\boldsymbol{y} = \frac{1}{2}\boldsymbol{x} - 3$. This always happens with inverse functions: The $\boldsymbol{x}$-coordinates and $\boldsymbol{y}$-coordinates of **all points** will be switched.

Try this: **Determine the function by finding two points in the given equation, switch the coordinates, and find the equation of the line through those points. Write your answer in function notation.**

1) $\boldsymbol{y} = 3\boldsymbol{x} + 5$

x	y	Point
0	5	(0, 5)
1		

FLIP THE COORDINATES: (______, ______) and (_______, ______)

INVERSE FUNCTION: $f^{-1}(x)$ = ________________

• Inverse Lines Using Points •

Determine the function by finding two points in the given equation, switch the coordinates, and find the equation of the line through those points. Write your answer in function notation.

1) $y = {}^{-}2x + 7$

x	y	Point
0	7	(0, 7)
1		

FLIP THE COORDINATES: (______, ______) and (_______, ______)

INVERSE FUNCTION: $f^{-1}(x) =$ __________________

2) $y = \frac{3}{2}x - 2$

x	y	Point

FLIP THE COORDINATES: (______, ______) and (_______, ______)

INVERSE FUNCTION: $f^{-1}(x) =$ __________________

3) $y = 5x - 3$

____

4) $f(x) = -\frac{1}{4}x + 1$

____

• Inverse Functions: Switching x and y •

Direct Teaching

Recall that the points in inverse functions are always switched. This means that the point (x, y) in the original function will always be (y, x) in the inverse function. This means that another way to find an inverse function is to switch the x and y values. Let's look at an example.

Example: Determine the inverse function of $y = {}^{-}2x + 4$.

Steps to Find the Inverse:

Step 1:
Switch the x and y in the equation.

$$y = {}^{-}2x + 4 \longrightarrow x = {}^{-}2y + 4$$

Step 2:
We have found the inverse function, but it is convention to rewrite the function so that the "new y" is isolated on one side of the equation.

$$x = {}^{-}2y + 4$$
$$x - 4 = {}^{-}2y$$
$$\frac{x-4}{{}^{-}2} = y$$

Step 3:
Rewrite the equation using inverse notation.

$$f^{-1}(x) = \frac{x-4}{{}^{-}2} = -\frac{1}{2}x + 2$$

Try these: **Find the inverse function of the given equation by switching x and y.**

1) $y = 7x + 1$

Switch: $x = 7y + 1$

Solve: ________________ $f^{-1}(x) =$ ________________

2) $y = {}^{-}3x - 5$

Switch: ________________

Solve: ________________ $f^{-1}(x) =$ ________________

• Inverse Functions: Switching x and y •

Find the inverse function of the given equation by switching *x* and *y*.

1) $y = 3x + 10$

Switch: $x = 3y + 10$

Solve: ________________ $f^{-1}(x) =$ ________________

2) $y = -\frac{1}{5}x + 1$

Switch: ________________

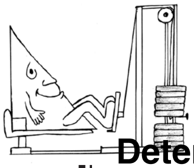
Solve: ________________ $f^{-1}(x) =$ ________________

3) $y = {}^{-}8x - 4$

____

4) $f(x) = \frac{2}{3}x - 6$

____



• Mastery Check: Inverse Functions •

Determine the inverse.

1) Determine the inverse of the function $y = 3x + 5$.

$f^{-1}(x) =$ ____________

2) Determine the inverse of the function $y = {}^{-}6x + 9$.

$f^{-1}(x) =$ ____________

3) Determine the inverse of the function $f(x) = \frac{1}{4}x - 3$.

$f^{-1}(x) =$ ____________

Challenge:

4) Determine the inverse of the function $f(x) = x^2 - 25$.

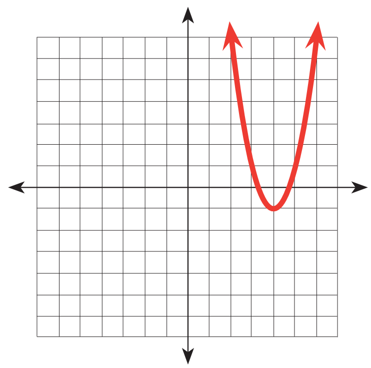
$f^{-1}(x) =$ ____________

• Equation of a Parabola from Vertex •

Direct Teaching

Usually, you need at least three points to determine the equation of a parabola. But if the vertex is one of the points, then we only need to find two points; and we can use *vertex form* to write the equation of the parabola. Let's look at an example.

EXAMPLE: Find the equation of the parabola.



The vertex is at (4, ⁻1). We can substitute that point into vertex form to get $y = a(x - 4)^2 - 1$.

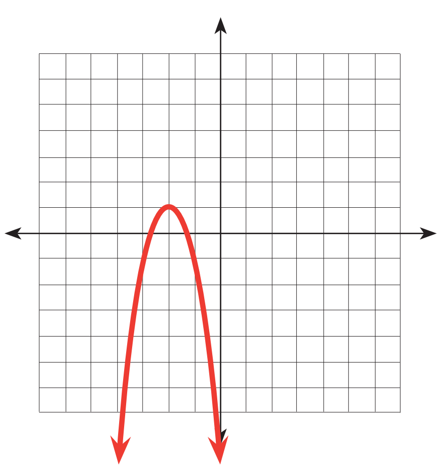
Now, we need to find a. We can find a by substituting a point on the graph into the equation and solving for a. The point (3, 1) is a point on the graph.

$$y = a(x - 4)^2 - 1$$
$$1 = a(3 - 4)^2 - 1$$
$$1 = a(^-1)^2 - 1$$
$$1 = a - 1$$
$$2 = a$$

So, $a = 2$. That means the equation of this parabola is $y = 2(x - 4)^2 - 1$.

***Try this*: Find the equation of the parabola.**

1)



VERTEX: (______, ______)

$y = a(x -$ ______$)^2 +$ ______

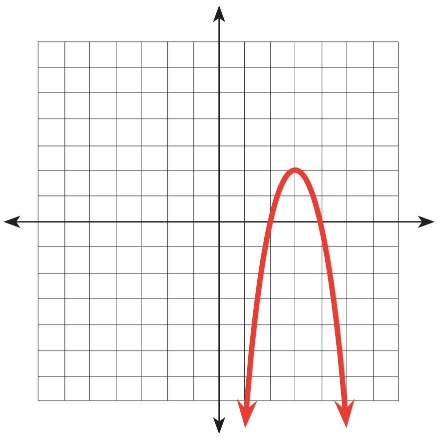
POINT FOR SUBSTITUTION: ________

$y =$ ________________________

• Equation of a Parabola from Vertex •

Find the equation of the parabola in vertex form.

1)



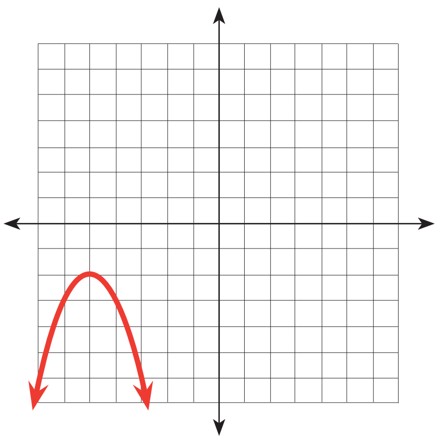
VERTEX: (______, ______)

$y = a(x - \_\_\_\_\_\_)^2 + \_\_\_\_\_\_$

POINT FOR SUBSTITUTION: ________

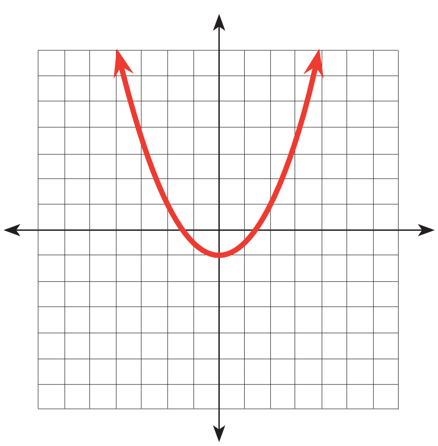
$y =$ ______________________________

2)



$y =$ ______________________________

3)



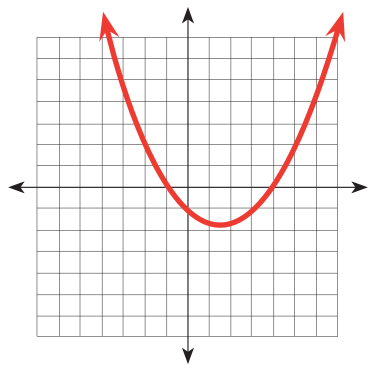
$y =$ ______________________________

• Equation of a Parabola from Roots •

Direct Teaching

Sometimes, the vertex is difficult to determine, so we would need to find three points. But, if two of those points are the roots of the parabola, then we can use the factored form of a parabolic equation, which is $y = a(x - r)(x - s)$, where r and s are the roots of the parabolic graph. We can use this form to find the equation from a graph.

Example: Find the equation of the parabola.



Here, we see that there is a root at $x = {}^{-}1$ and $x = 4$. So, we can substitute these values into $y = a(x - r)(x - s)$ to get $y = a(x + 1)(x - 4)$.

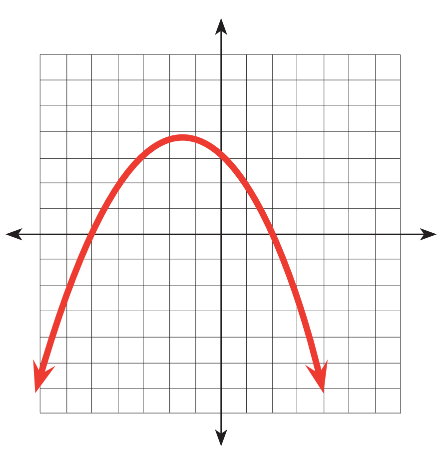
Now, we need to find a. We can find a by substituting a point on the graph into the equation and solve for a. The point (5, 2) can be used in this graph.

$$y = a(x + 1)(x - 4)$$
$$2 = a(5 + 1)(5 - 4)$$
$$2 = a(6)(1)$$
$$2 = 6a$$
$$\frac{1}{3} = a$$

So, $a = \frac{1}{3}$. That means the equation of this parabola in factored form is $y = \frac{1}{3}(x + 1)(x - 4)$.

Try this: **Find the equation of the parabola in factored form.**

1)



ROOTS: ______ and ______

$y = a(x -$ ______$)(x -$ ______$)$

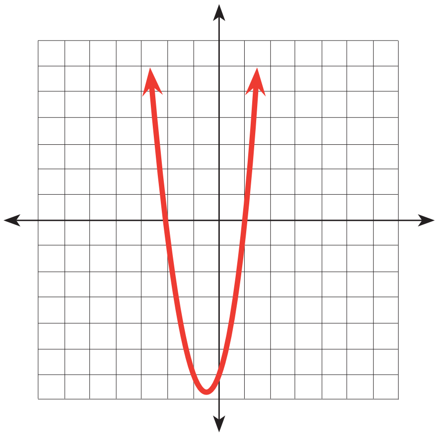
POINT FOR SUBSTITUTION: _______

$y =$ ________________________

• Equation of a Parabola from Roots •

Find the equation of the parabola in factored form.

1)



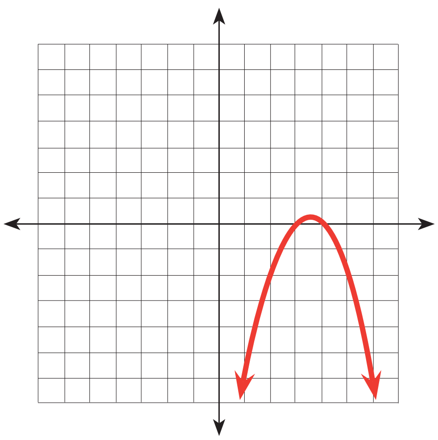
ROOTS: ______ and ______

$y = a(x -$ ______$)(x -$ ______$)$

POINT FOR SUBSTITUTION: ________

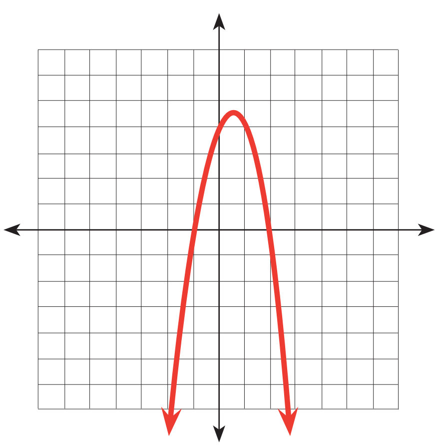
$y =$ ____________________________

2)

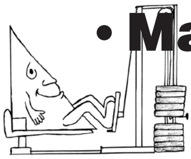


$y =$ ____________________________

3)



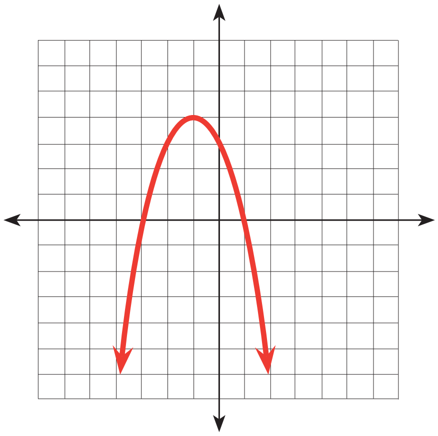
$y =$ ____________________________



• Mastery Check: Quadratic Equations from Graphs •

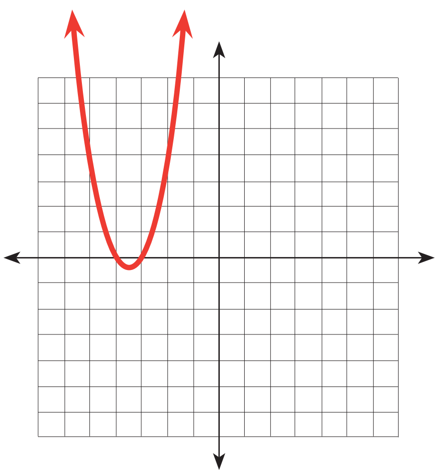
Find the equation of the parabola.

1)



$y =$ ______________________

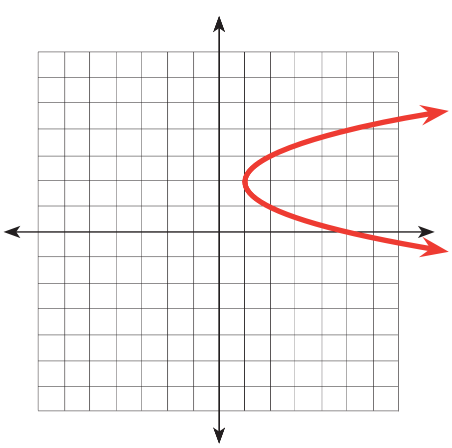
2)



$y =$ ______________________

Challenge:

3)



$x =$ ______________________

• Conjugates •

The CONJUGATE of a binomial is found by changing the sign between its terms.

Binomial: $(\boldsymbol{a} + \boldsymbol{b})$

Conjugate: $(\boldsymbol{a} - \boldsymbol{b})$

Product: $(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a} - \boldsymbol{b}) = (\boldsymbol{a}^2 - \boldsymbol{b}^2)$

If you multiply a binomial by its CONJUGATE, it results in a difference of squares.

EXAMPLE: Determine the conjugate and multiply the binomial by the conjugate.

$(\boldsymbol{x} + 5)$

The CONJUGATE is determined by changing the sign between the terms.

$(\boldsymbol{x} - 5)$

We can then take the product of the binomial and its conjugate to confirm that the result is a difference of squares.

$$(\boldsymbol{x} + 5)(\boldsymbol{x} - 5) = \boldsymbol{x}^2 + 5\boldsymbol{x} - 5\boldsymbol{x} - 25 = \boldsymbol{x}^2 - 25$$

Try these: **Determine the conjugate and multiply the binomial by the conjugate.**

1) $(\boldsymbol{x} + 3)$

CONJUGATE: ______________________

PRODUCT: ________________________

2) $(\boldsymbol{x} - 4)$

CONJUGATE: ______________________

PRODUCT: ________________________

3) $(\boldsymbol{x} - 1)$

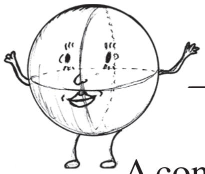
CONJUGATE: ______________________

PRODUCT: ________________________

4) $(\boldsymbol{x} + \boldsymbol{a})$

CONJUGATE: ______________________

PRODUCT: ________________________



• Polynomials with Radicals •

A conjugate can be especially useful when we have a binomial that contains a radical term. The result after calculating the product contains only rational terms.

Example: Multiply $(x + \sqrt{6})$ by its conjugate.

$$= (x + \sqrt{6})(x - \sqrt{6})$$

$$= x^2 - x\sqrt{6} + x\sqrt{6} - 6$$

$$= x^2 - 6$$

***Try these*: Multiply each binomial by its conjugate.**

1) $(x + \sqrt{2})$

CONJUGATE: ______________________

PRODUCT: ________________________

2) $(3 + \sqrt{x})$

CONJUGATE: ______________________

PRODUCT: ________________________

3) $(4 + \sqrt{5})$

____

4) $(x + \sqrt{14})$

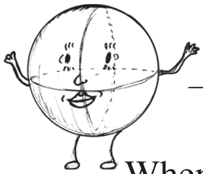
____

5) $(x + \sqrt{7})$

____

6) $(x + \sqrt{y})$

____



• Rationalizing the Denominator •

When we have a fraction that contains a radical in the denominator we often **RATIONALIZE THE DENOMINATOR**. Rationalizing the denominator is when we multiply the denominator by its conjugate to rid all radical terms from the denominator.

When we rationalize the denominator the value of the expression remains the same, but its form changes. This is similar to when we convert improper fractions to mixed numbers. Although the values are the same, the convention that is used requires the denominator to be rationalized.

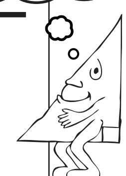
EXAMPLE: Simplify the rational expression.

$$\frac{2}{x - \sqrt{5}}$$

We need to rationalize the denominator. In order to do this we can multiply the top and bottom of the expression by the conjugate of the denominator.

$$\frac{2}{x - \sqrt{5}} \bullet \frac{x + \sqrt{5}}{x + \sqrt{5}} = \frac{2(x + \sqrt{5})}{(x - \sqrt{5})(x + \sqrt{5})} = \frac{2(x + \sqrt{5})}{x^2 - 5}$$

$$\frac{x + \sqrt{5}}{x + \sqrt{5}} = 1$$



Try these: **Simplify the rational expression.**

1) $\frac{1}{4 - \sqrt{3}} \bullet \frac{(4 + \sqrt{3})}{(4 + \sqrt{3})}$

____

2) $\frac{5}{10 + \sqrt{6}} \bullet \frac{(\quad - \sqrt{\quad})}{(\quad - \sqrt{\quad})}$

____

3) $\frac{10}{x - \sqrt{5}}$

____

4) $\frac{x}{x + \sqrt{7}}$

____

• Practice •

Simplify the rational expression.

1) $\frac{5}{2+\sqrt{7}}$ ______________

2) $\frac{x}{3+\sqrt{2}}$ ______________

3) $\frac{x}{x-2\sqrt{2}}$ ______________

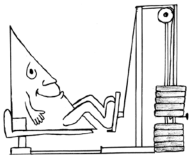
4) $\frac{8-\sqrt{3}}{5+\sqrt{3}}$ ______________

5) $\frac{s+1}{s-\sqrt{5}}$ ______________

6) $\frac{x-4}{1-\sqrt{7}}$ ______________

7) $\frac{x-\sqrt{2}}{y-\sqrt{2}}$ ______________

8) $\frac{x-\sqrt{5}}{x+\sqrt{5}}$ ______________



• Mastery Check: Conjugates • ☆

Determine the conjugate and multiply the binomial by the conjugate.

1) $(2 + \sqrt{3})$

CONJUGATE: ____________________

PRODUCT: ______________________

2) $(x - \sqrt{5})$

CONJUGATE: ____________________

PRODUCT: ______________________

Simplify the rational expression.

3) $\dfrac{5}{2 - \sqrt{5}}$

____

4) $\dfrac{7}{3 + \sqrt{3}}$

____

5) $\dfrac{x}{4 + \sqrt{7}}$

____

6) $\dfrac{x}{x - 3\sqrt{2}}$

____

Challenge: **Simplify the rational expression.**

7) $\dfrac{d}{x + r - \sqrt{2}}$

____

• Composition of Functions •

Use $f(x) = x^2 + 1$ and $g(x) = 2x - 1$ for the following.

1) $(f \circ g)(x) = \underline{f(g(x))} = \underline{f(2x-1)} = \underline{(2x-1)^2 + 1} = \_\_\_\_\_\_\_\_\_\_$

2) $(g \circ f)(x) = \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$

Find the composite function.

3) Determine $(f \circ g)(x)$ if $f(x) = 2x - 7$ and $g(x) = 2x^2 - 3$.

$(f \circ g)(x) =$ ________________ = ________________

EXPANDED | STANDARD FORM

4) Determine $(g \circ f)(x)$ if $f(x) = x^2 + 8$ and $g(x) = x^2 - 3$.

$(g \circ f)(x) =$ ________________ = ________________

EXPANDED | STANDARD FORM

5) Determine $(g \circ h)(x)$ if $g(x) = \frac{1}{2}x^2 - 2x$ and $h(x) = x + 2$.

$(g \circ h)(x) =$ ________________ = ________________

EXPANDED | STANDARD FORM

• Evaluating Composition of Functions •

Direct Teaching

We can evaluate a composition of functions using two methods: evaluating for each function separately or finding the composite function and then evaluating.

Example: Determine $(f \circ g)(1)$ if $f(x) = x^2 + 5x - 6$ and $g(x) = 4x - 1$.

Evaluate Separately	Composite Function
$f(g(1))$	$f(g(x))$
$g(1) = 4(1) - 1 = 3$	$f(4x - 1)$
$f(3) = (3)^2 + 5(3) - 6$	$(4x - 1)^2 + 5(4x - 1) - 6$
$9 + 15 - 6$	$16x^2 - 8x + 1 + 20x - 5 - 6$
18	$(f \circ g)(x) = 16x^2 + 12x - 10$
	$(f \circ g)(1) = 16(1)^2 + 12(1) - 10$
	$16 + 12 - 10$
	18

So, $(f \circ g)(1) = 18$.

If you are going to evaluate multiple values in a composite function, it is better to find the composite function first. That way, you already have the composite function ready for substituting the values. If you are only evaluating the composition at a single value, it may be easier to evaluate separately.

Try these: **Use $f(x) = x^2 - 5x + 8$ and $g(x) = 4x - 3$ to solve the following.**

1) $(f \circ g)(x)$ = __________ = ____________ = ______________________________ =

____

2) $(f \circ g)(2)$ = ______________

3) $(f \circ g)(^-1)$ = ______________

• Evaluating Composition of Functions •

☆

Use $f(x) = x + 1$ and $g(x) = 2x^2 - 5$ for the following.

1) $(f \circ g)(x) = \underline{f(g(x))} = \underline{f(2x^2 - 5)} = \underline{(2x^2 - 5) + 1} = \_\_\_\_\_\_\_\_\_\_$

2) $(g \circ f)(x) = \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

3) $(f \circ g)(4) = \_\_\_\_\_\_\_\_$

4) $(g \circ f)(1) = \_\_\_\_\_\_\_\_$

5) $(g \circ f)(7) = \_\_\_\_\_\_\_\_$

6) $(f \circ g)(3) = \_\_\_\_\_\_\_\_$

7) $(f \circ g)(^-5) = \_\_\_\_\_\_\_\_$

8) $(g \circ f)(y) = \_\_\_\_\_\_\_\_$

• Composition of Functions •

Use $f(x) = x^2 - 16$ and $g(x) = x^2 - 4x$ for the following.

1) $(f \circ g)(x) =$ ______________

2) $(f \circ g)(4) =$ __________

Use $f(x) = x^2 - x + 2$ and $g(x) = 2x^2 - 5$ for the following.

3) $(g \circ f)(x) =$ ______________

4) $(g \circ f)(^{-}1) =$ __________

---

5) If $g(1) = 4$ and $f(4) = 25$, then what is $(f \circ g)(1)$?

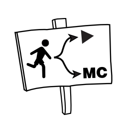
$(f \circ g)(1) =$ ________________

6) If $f(4) = 64$ and $g(64) = 100$, then what is $(g \circ f)(4)$?

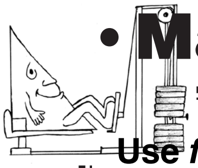
$(g \circ f)(4) =$ ________________

7) If $f(2) = 4$, $f(4) = 8$, $g(2) = 4$, and $g(4) = 2$, then what is $(f \circ g)(4)$?

$(f \circ g)(4) =$ ________________



Bypass



• Mastery Check: Composition of Functions • ☆

Use $f(x) = 4x - 3$, $g(x) = x^2 + 2$, and $h(x) = 2x^2 - 5x + 8$ to evaluate the following.

1) $(f \circ g)(x) =$ ______________

2) $(g \circ f)(x) =$ ______________

3) $(g \circ f)(1) =$ __________

4) $(f \circ g)(0) =$ __________

5) $(g \circ f)(3) =$ __________

6) $(f \circ g)\left(\frac{1}{2}\right) =$ __________

7) $(h \circ f)(^{-}2) =$ __________

8) $(f \circ f)(2) =$ __________

Challenge: Use the table to evaluate the functions.

x	$f(x)$	$g(x)$
0	2	2
1	3	0
2	6	2
3	11	8

9) $(f \circ g)(0) =$ ______________

10) $(g \circ f)(1) =$ ______________

• Solving Exponential Equations with the Same Base •

In an exponential equation, one way to solve the equation is to rewrite both sides to have the same base. Let's look at two examples.

Example 1: Solve $3^x = 81$.

First, notice that 81 is a power of 3. So we can rewrite 81 as 3^4.

$$3^x = 3^4$$

Since the bases are the same, the exponent expressions must be equivalent. So, $x = 4$.

Example 2: Solve $2^{3x-4} = 4$.

Let's rewrite 4 as a power of 2, then set the exponents equal to each other and solve.

$$2^{3x-4} = 2^2$$
$$3x - 4 = 2$$
$$3x = 6$$
$$x = 2$$

So, $x = 2$.

Try these: **Solve.**

1) $4^x = 16$

$4^x = 4^{\square}$

$x =$ ______________

2) $5^{2x+5} = 125$

$5^{2x+5} = 5^{\square}$

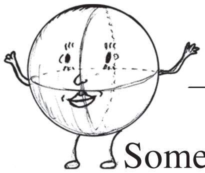
$x =$ ______________

3) $2^{3y} = 512$

$y =$ ______________

4) $10^{a-2} = 1{,}000$

$a =$ ______________



• Solving Exponential Equations •

Sometimes both terms have to be rewritten to have the same base.

Example: Solve $16^{2x+6} = 64^{3x-1}$.

While 64 is not a power of 16, both 64 and 16 are powers of 4. So we can rewrite both sides of the equation as powers of 4.

$$16^{2x+6} = 64^{3x-1}$$

$$(4^2)^{(2x+6)} = (4^3)^{(3x-1)}$$

$$2(2x+6) = 3(3x-1)$$

$$4x + 12 = 9x - 3$$

$$15 = 5x$$

$$3 = x$$

So, $x = 3$.

Note that 16 and 64 are also powers of 2, so we could have rewritten the equation as $(2^4)^{(2x+6)} = (2^6)^{(3x-1)}$. We get the same solution either way: $x = 3$.

Try these: **Solve.**

1) $4^x = 8^{x-1}$

4 and 8 are both powers of ______.

$x =$ ______________

2) $9^{x-1} = 27^{3x+4}$

9 and 27 are both powers of ______.

$x =$ ______________

3) $1{,}000^{3w-2} = 100{,}000^{w+4}$

$w =$ ______________

4) $81^{k-3} = 9^{4k+6}$

$k =$ ______________

• Exponential Equations with Rational Bases • ☆

Example: Solve $\left(\frac{1}{16}\right)^{x+4} = 64^{x-1}$.

$$\left(\frac{1}{16}\right)^{x+4} = 64^{x-1}$$

$$(4^{-2})^{(x+4)} = (4^{3})^{(x-1)}$$

$$^{-}2(x+4) = 3(x-1)$$

$$^{-}2x - 8 = 3x - 3$$

$$^{-}5 = 5x$$

$$^{-}1 = x$$

So, $x = {}^{-}1$.

Try these: **Solve.**

1) $\left(\frac{1}{9}\right)^{2x-9} = 27^{2x-4}$

$\frac{1}{9}$ and 27 are both powers of ______.

$x =$ ______________

2) $\left(\frac{1}{6}\right)^{3x+3} = \left(\frac{1}{36}\right)^{-2x+1}$

$\frac{1}{6}$ and $\frac{1}{36}$ are both powers of ______.

$x =$ ______________

3) $625 = 25^{p-2}$

$p =$ ______________

4) $4^{d-14} = \left(\frac{1}{8}\right)^{3d+2}$

$d =$ ______________

• Inconsistent Equations •

Sometimes we may get an inconsistent solution.

Example: Solve $16^{3x+1} = 8^{4x+3}$.

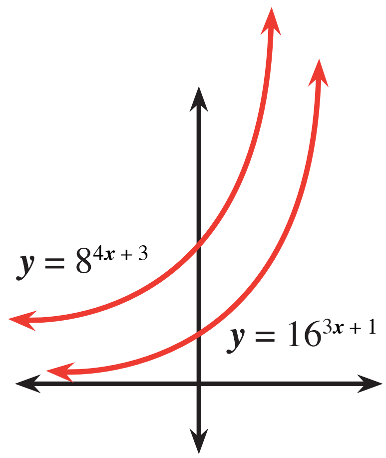
$$16^{3x+1} = 8^{4x+3}$$

$$(2^4)^{(3x+1)} = (2^3)^{(4x+3)}$$

$$4(3x + 1) = 3(4x + 3)$$

$$12x + 4 = 12x + 9$$

$$4 \stackrel{?}{=} 9$$



We have an inconsistent solution. Since 4 cannot equal 9, there is no solution. We can see from that graph that the solution is inconsistent because the graphs do not intersect.

If the last line is always true, such as 5 = 5, then there are infinitely many solutions. The graphs of both expressions would be the same.

Try these: **Solve. If the solution is inconsistent, write "inconsistent."**

1) $7^{3x-3} = 343^{x-3}$

$x =$ ______________

2) $216^{x} = 36^{x+3}$

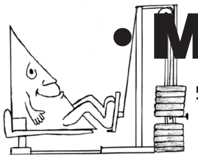
$x =$ ______________

3) $9^{-4t+4} = \left(\frac{1}{81}\right)^{2t-2}$

$t =$ ______________

4) $5^{9m-3} = 125^{3m+2}$

$m =$ ______________



• Mastery Check: Exponential Equations •

Solve. If the solution is inconsistent, write "inconsistent."

1) $3^{3x+1} = 9^{2x-5}$

$x =$ ______________

2) $25^{x+5} = 125^{-3x+2}$

$x =$ ______________

3) $100^{5y+3} = 100{,}000^{2y+7}$

$y =$ ______________

4) $\left(\frac{1}{4}\right)^{4n-3} = \left(\frac{1}{8}\right)^{n+3}$

$n =$ ______________

Challenge:

5) $(0.01)^{2x-3} = 1{,}000^{x+9}$

$x =$ ______________

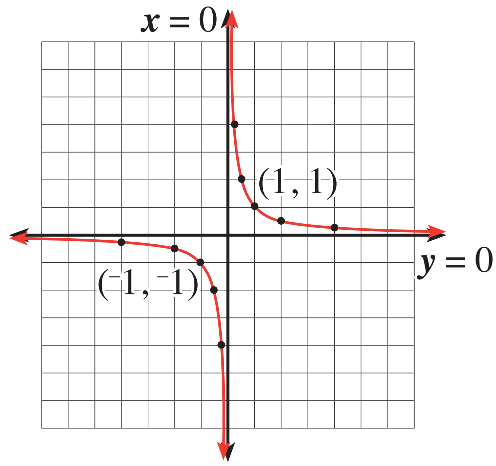
• Sketching Rational Functions •

Direct Teaching

Let's take a look at a few points that are satisfied to help us sketch the following rational parent function.

Example: Sketch the function, $f(x) = \frac{1}{x}$, by finding key points.

x	y
$^{-}4$	$-\frac{1}{4}$
$^{-}2$	$-\frac{1}{2}$
$^{-}1$	$^{-}1$
$-\frac{1}{2}$	$^{-}2$
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$



The rational parent function does not have an x– or y– intercept, and it consists of non–continuous parts in quadrants I and III. Rational functions are defined by having both vertical and horizontal asymptotes, and the denominator of a rational function also cannot equal 0, otherwise the function is undefined.

Based on the sketch, we can tell that the function gets close to the line $x = 0$ from both the left and right side, but as it cannot equal 0. So, $x = 0$ is the **vertical asymptote**. Because 0 is not an included value, the domain of the rational parent function is $(^{-}\infty, 0) \cup (0, \infty)$.

As the value of x gets very large or very small, the value of the function again gets really close to 0. Here, $y = 0$ is the **horizontal asymptote**. The range of the rational parent function is $(^{-}\infty, 0) \cup (0, \infty)$.

Try this:

1) How does the fact that there are now two asymptotes, which are perpendicular to each other, affect the sketch of the parent function?

____

• Function Transformations That Affect *x* •

Rational functions can be sketched using transformations. Let's first take a look at the transformations that affect ***x*** directly in a rational function.

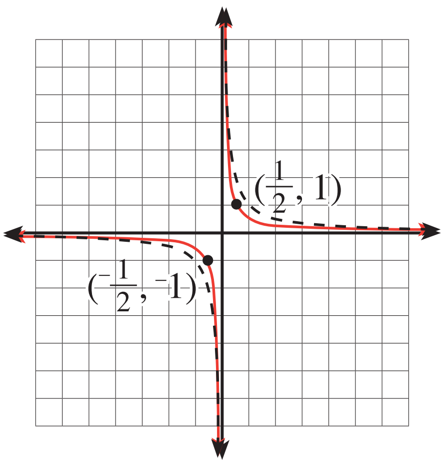
Direct Teaching

Example: Sketch $f(\boldsymbol{x}) = \dfrac{1}{^-2(\boldsymbol{x} - 3)}$

Here, the transformation of the function is $f(^-2(\boldsymbol{x} - 3))$, and the parent function is $\dfrac{1}{\boldsymbol{x}}$.

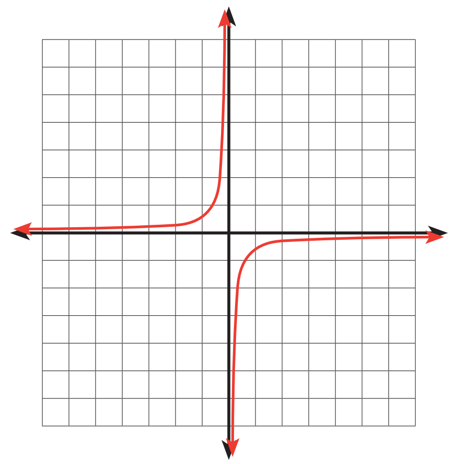
Step 1: $f(\boldsymbol{x}) = \dfrac{1}{2\boldsymbol{x}}$

The parent function is horizontally compressed by a factor of $\frac{1}{2}$.



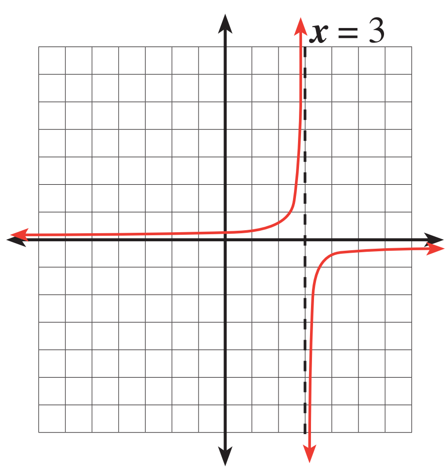
Step 2: $f(\boldsymbol{x}) = \dfrac{1}{^-2\boldsymbol{x}}$

The function is now reflected over the ***y***–axis.



Step 3: $f(\boldsymbol{x}) = \dfrac{1}{^-2(\boldsymbol{x} - 3)}$

The final transformation that affects ***x*** is a horizontal shift right 3 units. This shifts the **vertical asymptote**.



• Function Transformations •

Describe the function transformations and their order.

1) $f(^{-}x)$

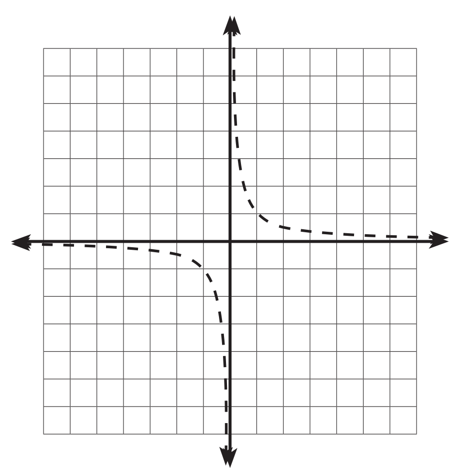
2) $f(^{-}x + 7)$

3) $f(\frac{1}{2}x + 5)$

4) $f(^{-}4x + 12)$

Sketch the function.

5) $f(x) = \dfrac{1}{^{-}4(x + 1)}$



• Function Transformations That Affect $f(x)$ •

Direct Teaching

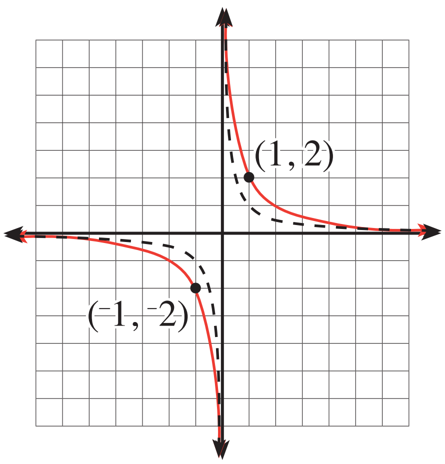
Next, let's take a look at the transformations that affect $f(x)$ directly in a rational function.

EXAMPLE: Sketch $f(x) = \frac{^-2}{x} + 1$

Here, the transformation of the function is $^-2f(x) + 1$, and the parent function is $\frac{1}{x}$.

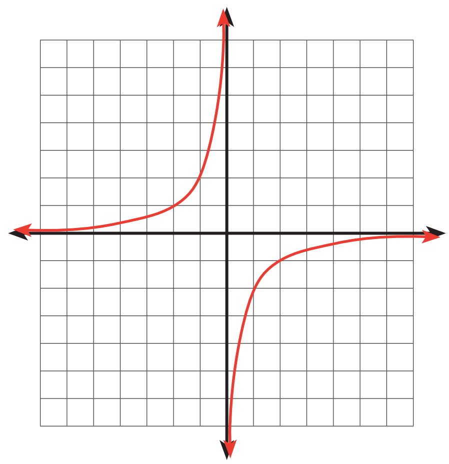
STEP 1: $f(x) = \frac{2}{x}$

The parent function is vertically stretched by a factor of 2.



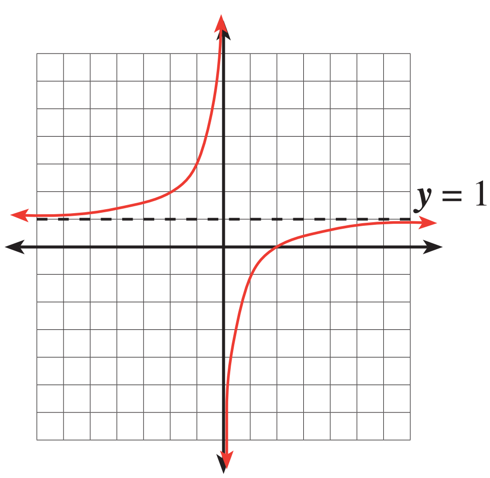
STEP 2: $f(x) = \frac{^-2}{x}$

The function is now reflected over the x–axis.



STEP 3: $f(x) = \frac{^-2}{x} + 1$

The final transformation is a vertical shift up 1 unit. This shifts the **HORIZONTAL ASYMPTOTE**.



• Function Transformations •

☆

Describe the function transformations and their order.

1) $2f(x) + 1$

____

____

2) $^{-}f(x) - 3$

____

____

____

3) $^{-}5f(x) + 4$

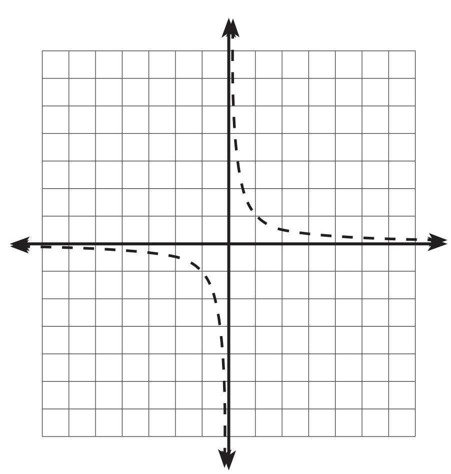
____

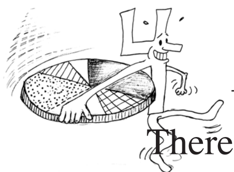
____

____

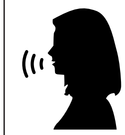
Sketch the function.

4) $f(x) = \frac{^{-}3}{x} - 2$





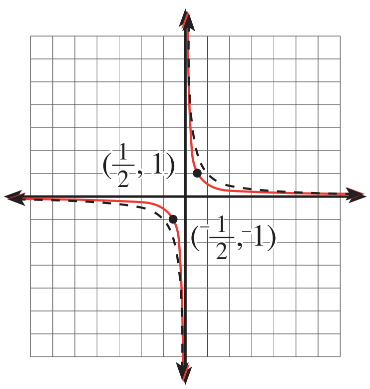
• Sketching Rational Functions •



There are six possible transformations that can be made to the rational parent function, and these transformations need to be made in a certain order. Let's look at an example.

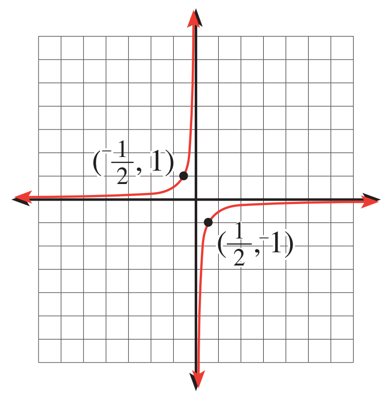
Example: $f(x) = \left(\dfrac{5}{^{-}2(x-3)}\right) + 2$

Step 1: The parent function is horizontally compressed by a factor of 2.



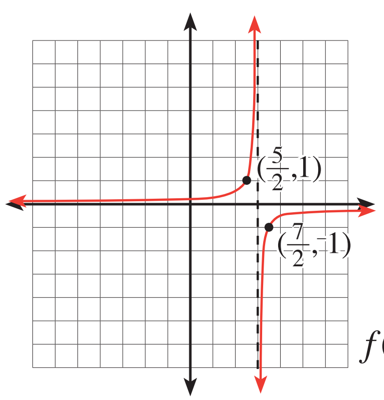
$f(x) = \dfrac{1}{2x}$

Step 2: The function is now reflected over the ***y***–axis.



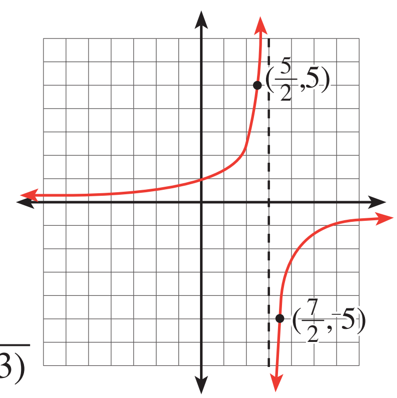
$f(x) = \dfrac{1}{^{-}2x}$

Step 3: The next transformation is a horizontal shift right 3 units.



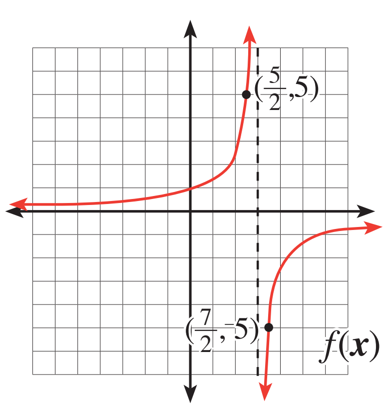
$f(x) = \dfrac{1}{^{-}2(x-3)}$

Step 4: The parent function is vertically stretched by a factor of 5.



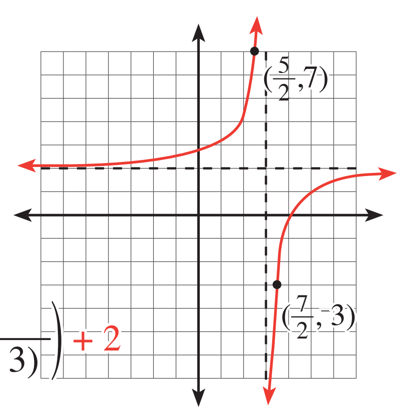
$f(x) = \dfrac{5}{^{-}2(x-3)}$

Step 5: There is no reflection over the ***x***–axis.



$f(x) = \left(\dfrac{5}{^{-}2(x-3)}\right)$

Step 6: The final transformation is a vertical shift up 2 units.



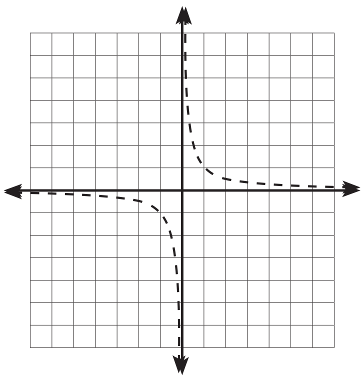
$f(x) = \left(\dfrac{5}{^{-}2(x-3)}\right) + 2$

• Sketching Rational Functions •

Sketch and describe the function transformation by highlighting each of the particular transformations. Note that you may not need to use all six steps.

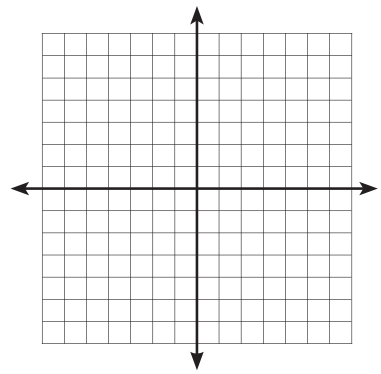
1) $f(x) = {}^{-}\left(\dfrac{1}{3(x+2)}\right) - 4$

Step 1: Horizontal compression by a factor of $\frac{1}{3}$.



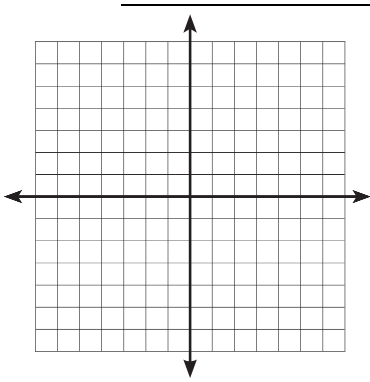
$f(x) = \dfrac{1}{3x}$

Step 2: ______________________________



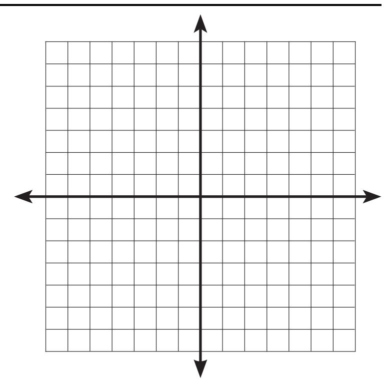
$f(x) =$

Step 3: ______________________________



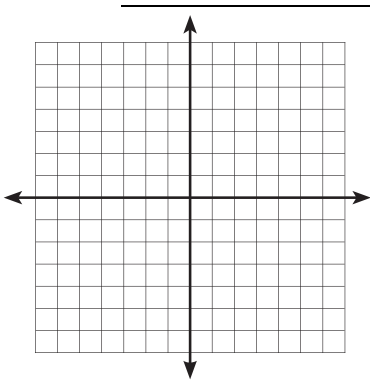
$f(x) =$

Step 4: ______________________________



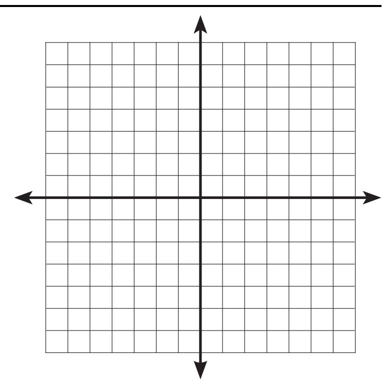
$f(x) =$

Step 5: ______________________________

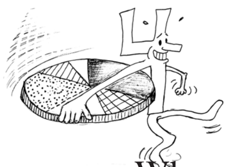


$f(x) =$

Step 6: ______________________________



$f(x) =$



• Sketching Rational Functions •

Direct Teaching

When sketching a rational function, we can use a few key features to make this process more efficient. Let's first determine the asymptotes. These lines can be determined from the values of b and d.

$$\left(\frac{c}{a(\boldsymbol{x} - b)}\right) + d$$

If the parent function is shifted, there will be an $\boldsymbol{x}$–intercept, a $\boldsymbol{y}$–intercept, or both. This gives us up to two points on the sketch.

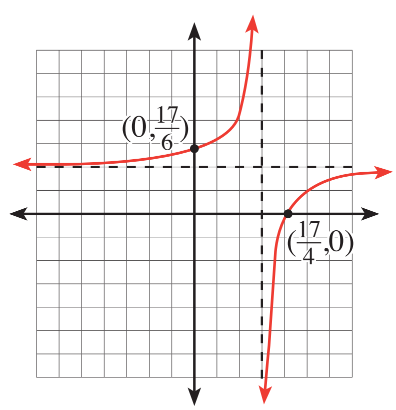
We can also test the $\boldsymbol{x}$–value of a point in each section separated by the vertical asymptote. This will let us know if the function is above or below the horizontal asymptote in any given section.

Example: Determine the horizontal asymptote, the vertical asymptote, and find points to sketch the rational function.

$$f(\boldsymbol{x}) = \left(\frac{5}{^{-}2(\boldsymbol{x} - 3)}\right) + 2$$

The horizontal asymptote is determined by the value of d, and is found to be $\boldsymbol{y} = 2$. Similarly, the vertical asymptote is determined by b, here it is $\boldsymbol{x} = 3$.

The $\boldsymbol{x}$–intercept is then found to be the point $(\frac{17}{4}, 0)$, and the $\boldsymbol{y}$–intercept is found to be the point $(0, \frac{17}{6})$. Now notice, this is the same example we sketched earlier using transformations.



Here, the $\boldsymbol{x}$– and $\boldsymbol{y}$– intercepts are on different sides of the vertical asymptote, so we can use these two points and the asymptotes to sketch the function.

If the $\boldsymbol{x}$– and $\boldsymbol{y}$– intercepts are on the same side of the vertical asymptote, we need to find one additional point to sketch the function.

• Sketching Rational Functions •

Sketch the following function. Label key points and the asymptotes on the graph.

1) $f(\boldsymbol{x}) = \left(\dfrac{1}{^{-}2(\boldsymbol{x} + 4)}\right) - 1$

What is the horizontal shift? ____________________________

What is the equation of the vertical asymptote? ____________________

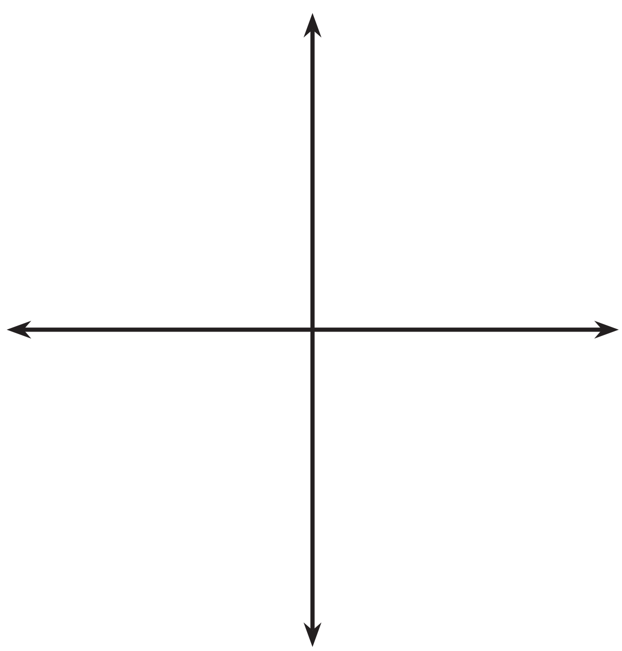
What is the vertical shift? ______________________________

What is the equation of the horizontal asymptote? __________________

What is the ***x***–intercept? ________________

What is the ***y***–intercept? ________________

Sketch the curves such that they approach the asymptotes.



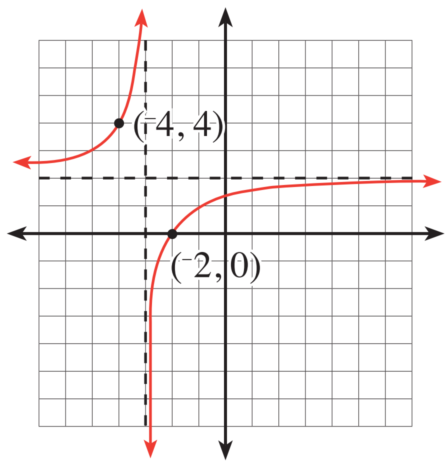
• Sketching Rational Functions •

Direct Teaching

If the x– and/or y–intercepts cannot be found, we can use other points as long as we have a point on either side of the vertical asymptote.

$$f(x) = \left(\frac{4}{{}^{-}2(x+3)}\right) + 2$$

Looking at the function above, we know the vertical asymptote is $x = {}^{-}3$. So, let's find the value of the function on either side of this line. These points are found to be $({}^{-}4, 4)$ and $({}^{-}2, 0)$. Using these two points and the asymptotes, we can sketch the function.



Sketch the following function. Label key points and the asymptotes.

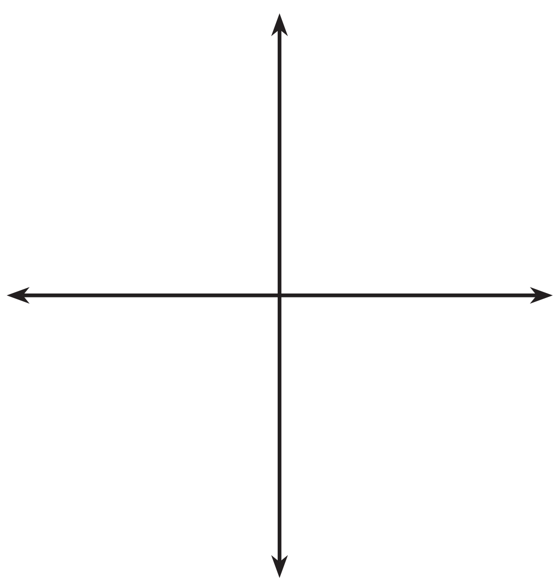
1) $f(x) = {}^{-}\left(\frac{4}{(x+2)}\right) + 3$

What is the equation of the vertical asymptote? ____________________

What is the equation of the horizontal asymptote? ____________________

Identify one point on each side of the vertical asymptote? ________ ________

Sketch the curves such that they approach the asymptotes.



• Sketching Rational Functions •

Sketch the following function. Label key points and the asymptotes.

1) $f(\boldsymbol{x}) = {}^{-}\left(\frac{9}{{}^{-}2\boldsymbol{x}}\right) + 2$

What is the equation of the vertical asymptote? ______________________

What is the equation of the horizontal asymptote? ____________________

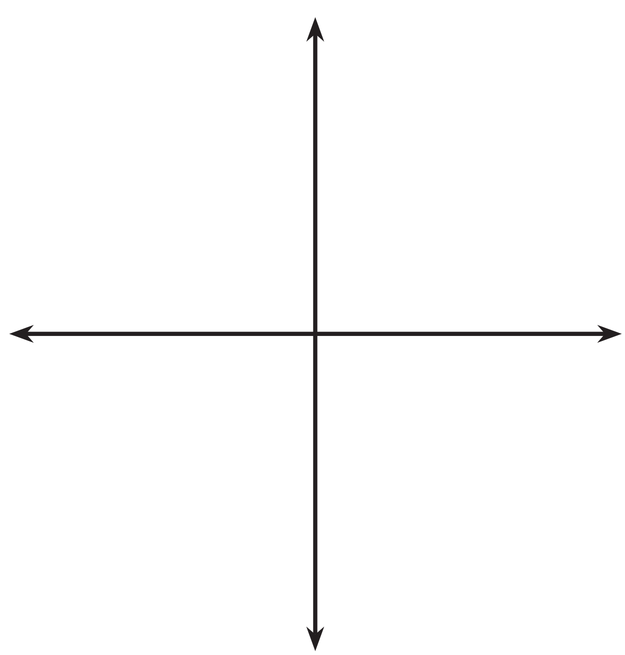
What is the ***x***–intercept? ___________________

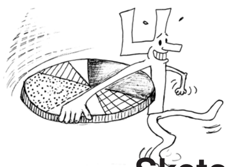
What is the ***y***–intercept? ___________________

Do we need an additional point to sketch the function? ________________

If so, what point can we use? ____________________________________

Sketch the curves such that they approach the asymptotes.



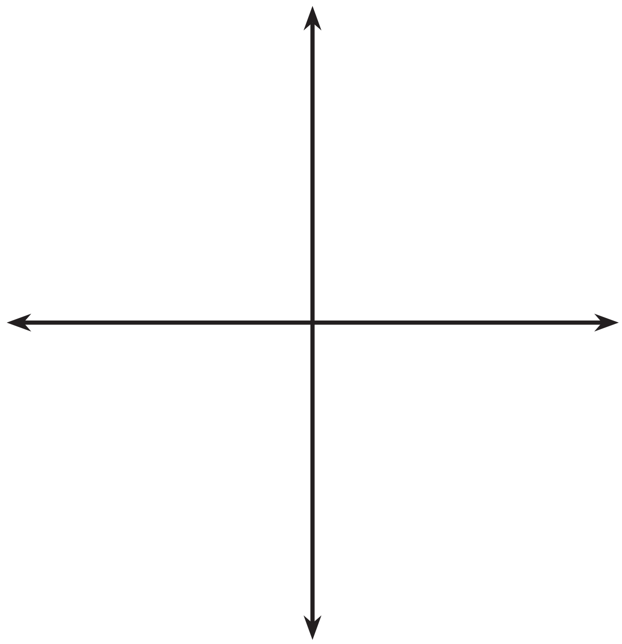


• Sketching Rational Functions •

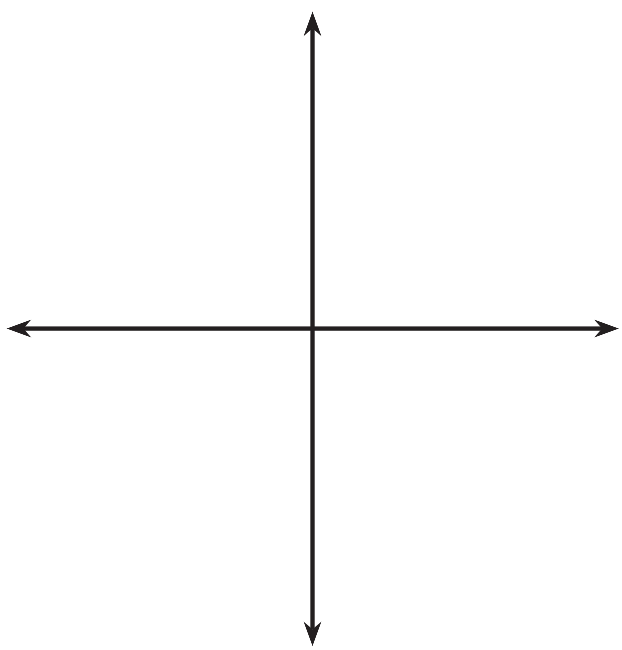
☆

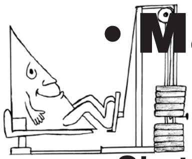
Sketch the function. Determine the vertical and horizontal asymptotes. Label the asymptotes on the graph.

1) $f(x) = {}^{-}\left(\dfrac{3}{{}^{-}2(x - 3)}\right)$



1) $f(x) = {}^{-}\left(\dfrac{4}{2(x + 1)}\right) + 1$



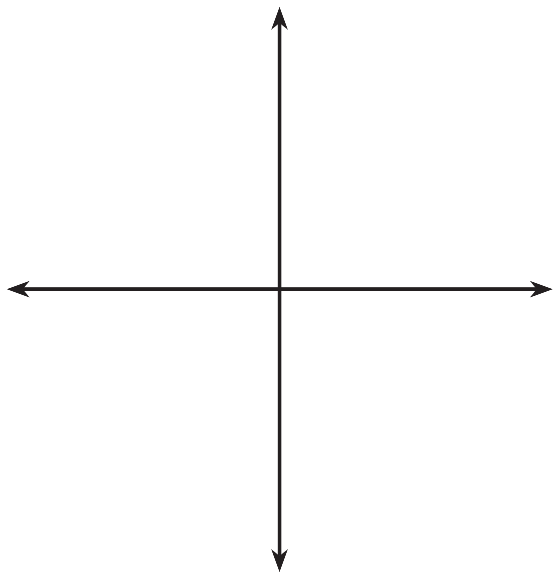


• Mastery Check: Sketching Rational Functions •

☆

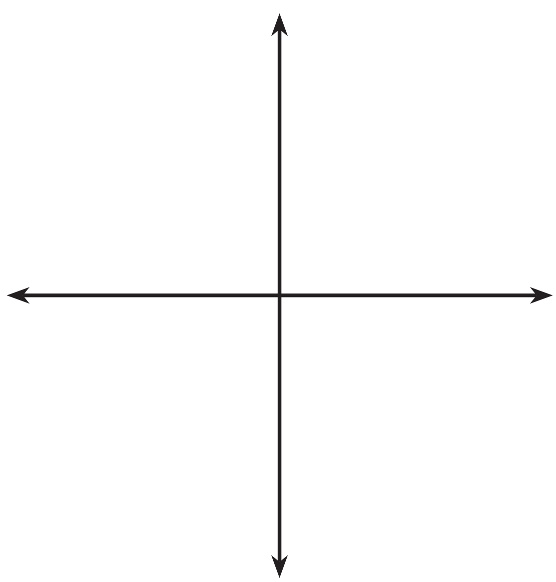
Sketch the function. Determine the vertical and horizontal asymptotes. Label the asymptotes on the graph.

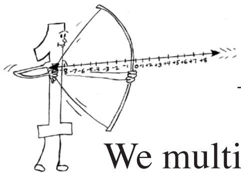
1) $f(x) = \dfrac{2}{x + 1} - 3$



Challenge:

2) $f(x) = \dfrac{2}{(x - 1)^2} - 4$





• Multiply Rational Expressions •

Direct Teaching

We multiply rational expressions the same way we multiply numeric fractions: we multiply the numerators, then multiply the denominators. If the expressions are polynomials, we write them in factored form [$(x + a)(x + b)$], if possible, instead of using the distributive property to write them in standard form [$ax^2 + bx + c$].

Example: Simplify $\dfrac{x - 1}{x^2 - 5x + 6} \bullet \dfrac{x^2 + 6x + 8}{x^2 - 11x + 30}$.

First, let's write the polynomials in factored form.

$$\frac{x - 1}{x^2 - 5x + 6} \bullet \frac{x^2 + 6x + 8}{x^2 - 11x + 30} = \frac{(x - 1)}{(x - 2)(x - 3)} \bullet \frac{(x + 2)(x + 4)}{(x - 5)(x - 6)}$$

After writing the terms in factored form, we combine the numerators and denominators.

$$\frac{(x - 1)}{(x - 2)(x - 3)} \bullet \frac{(x + 2)(x + 4)}{(x - 5)(x - 6)} = \frac{(x - 1)(x + 2)(x + 4)}{(x - 2)(x - 3)(x - 5)(x - 6)}$$

We leave our solution in factored form. That is considered simplified form.

Try these: **Simplify the following.**

1) $\dfrac{x - 10}{x^2 - 9x + 8} \bullet \dfrac{x^2 - 12x - 13}{x + 7}$

FACTORED: $\dfrac{x - 10}{(x - 1)(x - 8)} \bullet$ ________________

SIMPLIFIED: ________________________

2) $\dfrac{x^2 + 3x + 2}{x + 15} \bullet \dfrac{x - 4}{3x^2 - x - 2}$

FACTORED: ________________ • ________________

SIMPLIFIED: ________________________

• Divide Rational Expressions •

☆

Direct Teaching

We divide rational expressions the same way we divide fractions: we invert and multiply. If the expressions are polynomials, it is beneficial to write the polynomials in factored form.

Example: Simplify $\dfrac{x^2 + 6x + 5}{x^2 - 6x + 8} \div \dfrac{x + 9}{2x^2 + 7x + 3}$.

First let's write the polynomials in factored form.

$$\frac{x^2 + 6x + 5}{x^2 - 6x + 8} \div \frac{x + 9}{2x^2 + 7x + 3} = \frac{(x + 1)(x + 5)}{(x - 2)(x - 4)} \div \frac{(x + 9)}{(2x + 1)(x + 3)}$$

After writing the terms in factored form, we invert and multiply.

$$\frac{(x + 1)(x + 5)}{(x - 2)(x - 4)} \div \frac{(x + 9)}{(2x + 1)(x + 3)} = \frac{(x + 1)(x + 5)}{(x - 2)(x - 4)} \bullet \frac{(2x + 1)(x + 3)}{(x + 9)}$$

$$= \frac{(x + 1)(x + 5)(2x + 1)(x + 3)}{(x - 2)(x - 4)(x + 9)}$$

So, $\dfrac{x^2 + 6x + 5}{x^2 - 6x + 8} \div \dfrac{x + 9}{2x^2 + 7x + 3}$ simplifies to $\dfrac{(x + 1)(x + 5)(2x + 1)(x + 3)}{(x - 2)(x - 4)(x + 9)}$.

***Try these*:** **Simplify the following.**

1) $\dfrac{x^2 - 5x + 4}{x^2 - 25} \div \dfrac{x^2 + 10x}{x + 9}$

FACTORED:

$\dfrac{(x - 1)(x - 4)}{(x - 5)(x + 5)} \div$ ______________

INVERT AND MULTIPLY:

$\dfrac{(x - 1)(x - 4)}{(x - 5)(x + 5)} \bullet$ ______________

SIMPLIFIED:

____

2) $\dfrac{x^2 + 7x + 12}{x + 15} \div \dfrac{x - 4}{2x^2 + x - 3}$

FACTORED:

______________ $\div$ ______________

INVERT AND MULTIPLY:

______________ $\bullet$ ______________

SIMPLIFIED:

____

• Multiply and Divide Rational Expressions •

☆

Simplify the following fractions.

1) $\dfrac{x^2 + 5x + 4}{x^2 - 4} \bullet \dfrac{x^2 + 7x}{x - 3}$

FACTORED: ______________ • ______________

SIMPLIFIED:

____

2) $\dfrac{y^2 + 8y + 15}{3y^2 - y - 2} \div \dfrac{y + 8}{y^2 + y}$

FACTORED: ______________ ÷ ______________

INVERT AND MULTIPLY:

______________ • ______________

SIMPLIFIED:

____

3) $\dfrac{x^2 - 16}{x^2 + 16} \div \dfrac{x^2 + 9}{x^2 - 9}$

SIMPLIFIED:

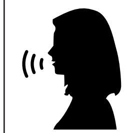
____

4) $\dfrac{a^3 - 15a^2 + 56a}{a - 20} \bullet \dfrac{a^2 - 81}{5a^2 + 14a + 8}$

SIMPLIFIED:

____

• Simplifying Rational Expressions •



Direct Teaching

When we have a product or a quotient of rational expressions, we can simplify further if there is a term in the numerator that matches a term in the denominator.

EXAMPLE: Simplify $\dfrac{x-3}{x^2+x-12} \bullet \dfrac{x^2-16}{x^2-9x+20}$.

First, let's write the polynomials in factored form and then multiply.

$$\frac{x-3}{x^2+x-12} \bullet \frac{x^2-16}{x^2-9x+20} = \frac{(x-3)}{(x-3)(x+4)} \bullet \frac{(x-4)(x+4)}{(x-4)(x-5)}$$

As long as the same term is on the top and bottom of a rational expression, it can be reduced.

$$\frac{\cancel{x-3}}{\cancel{(x-3)}\cancel{(x+4)}} \bullet \frac{\cancel{(x-4)}\cancel{(x+4)}}{\cancel{(x-4)}(x-5)} = \frac{1}{(x-5)}$$

Try these: **Simplify the following fractions.**

1) $\dfrac{x^2-3x-4}{x^2-16} \bullet \dfrac{x^2+4x}{x-2}$

FACTORED: $\dfrac{(x+1)(x-4)}{(x-4)(x+4)} \bullet$ ______________

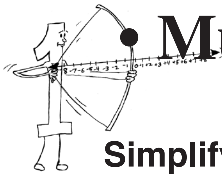
SIMPLIFIED: ______________

2) $\dfrac{x^2-7x+12}{x+1} \div \dfrac{x-4}{x^2-2x-3}$

FACTORED: ______________ ÷ ______________

INVERT AND MULTIPLY: ______________ • ______________

SIMPLIFIED: ______________



• Multiply and Divide Rational Expressions • ☆

Simplify the following fractions.

1) $\dfrac{x^2 - 3x - 4}{x^2 - 16} \bullet \dfrac{x^2 + 4x}{x - 2}$

FACTORED: ________________ • ________________

SIMPLIFIED:

____

2) $\dfrac{p^2}{2p^2 + 5p + 3} \div \dfrac{7p^2 - 3p}{4p^2 + 4p + 1}$

SIMPLIFIED:

____

3) $\dfrac{x^2 + 5x + 6}{x^2 + 5x - 6} \bullet \dfrac{x^2 - 5x + 6}{x^2 - 5x - 6}$

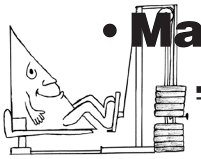
SIMPLIFIED:

____

4) $\dfrac{c^3 - 13c^2 + 42c}{c - 6} \div \dfrac{3c^2 + c}{9c^2 + 12c + 4}$

SIMPLIFIED:

____



• Mastery Check: Multiply and Divide Rational Expressions •

Simplify the following fractions.

1) $\dfrac{x^2 + 7x + 10}{x} \bullet \dfrac{x^2}{x^2 + 2x - 15}$

SIMPLIFIED: ____________________

2) $\dfrac{x^2 + 8x + 15}{x^2 - 7x + 12} \div \dfrac{x + 5}{x^2 - 3x - 18}$

SIMPLIFIED: ____________________

3) $\dfrac{2n^2 + 3n + 1}{n^2 - 3n + 2} \div \dfrac{6n^2 + 5n + 1}{n^2 - 5n + 4}$

SIMPLIFIED: ____________________

Challenge:

4) $\dfrac{z^3 - 27}{z^2 - z - 6} \div \dfrac{z^4 - 16}{z^4 - 5z^2 + 4}$

SIMPLIFIED: ____________________

• Least Common Multiple of Factored Polynomials •

When we find the **LEAST COMMON MULTIPLE (LCM)** of polynomials, we list the factors of the polynomials. Then we use those factors to determine the **LCM**.

EXAMPLE 1: Find the Least Common Multiple of $(x + 1)(x - 2)$ and $(x - 2)(x + 3)$.

When finding the **LCM** of polynomials, they should be in factored form. These both have $(x - 2)$ as a factor, so that term is used once in the **LCM**.

So, the **LCM** of $(x + 1)(x - 2)$ and $(x - 2)(x + 3)$ is $(x + 1)(x - 2)(x + 3)$.

EXAMPLE 2: Find the Least Common Multiple of $(x - 3)^3(x + 1)$ and $(x - 3)(x + 1)^2$.

Similar to finding the **LCM** of monomials, the **LCM** of polynomials always contains the largest of the exponents for each term.

So, the **LCM** of $(x - 3)^3(x + 1)$ and $(x - 3)(x + 1)^2$ is $(x - 3)^3(x + 1)^2$.

Try these: **Determine the LCM for the polynomials.**

1) $(y - 5)(y - 7)$ and $(y - 7)(y - 8)$

____

2) $(a + 3)^5(a + 1)$ and $(a + 3)^8(a + 1)^4$

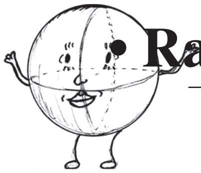
____

3) $(x - 3)(x + 4)$ and $(y - 3)(y + 4)$

____

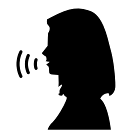
4) $(n + 5)(n + 6)$ and $n^2 + 7n + 6$

____



• Rational Expressions with Binomial Denominators • ☆

The Law of SAMEness



Direct Teaching

We use the **Least Common Denominator (LCD)** to simplify rational expressions.

Example: Simplify $\frac{2}{x+1} + \frac{4}{x-3}$.

Steps to Simplify:

Step 1:
Find the **LCD**.

The **LCD** of $x + 1$ and $x - 3$ is $(x + 1)(x - 3)$.

Step 2:
Rewrite with equivalent fractions so that the fractions have the same name.

$$\frac{2}{x+1} + \frac{4}{x-3} =$$
$$\frac{2x-6}{(x+1)(x-3)} + \frac{4x+4}{(x+1)(x-3)}$$

Step 3:
Simplify the fractions.

$$\frac{2x-6}{(x+1)(x-3)} + \frac{4x+4}{(x+1)(x-3)} =$$
$$\frac{6x-2}{(x+1)(x-3)}$$

So, $\frac{2}{x+1} + \frac{4}{x-3} = \frac{6x-2}{(x+1)(x-3)}$.

Try these: **Simplify by using the LCD.**

1) $\frac{1}{x-4} + \frac{3}{x+2}$

LCD = ________________

____

2) $\frac{1}{x+10} + \frac{5x}{x-9}$

LCD = ________________

____

• Rational Expressions with Binomial Denominators • ☆

Simplify by using the LCD.

1) $\dfrac{x}{x+7} + \dfrac{5}{x-2}$

LCD = ________________

____

2) $\dfrac{3}{x-3} - \dfrac{7x}{x+1}$

LCD = ________________

____

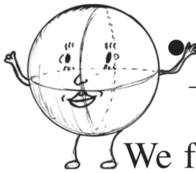
3) $\dfrac{x-7}{x+4} - \dfrac{x^2+x}{x-1}$

____

Extending Knowledge

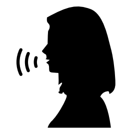
4) $\dfrac{3x+2}{x-1} + \dfrac{5x}{1-x}$

____



•Adding and Subtracting Rational Expressions •

The Law of SAMEness



Direct Teaching

We factor the denominators when finding the LCD to ensure our final answer will be reduced.

Try this: **Fill in the blanks in the example.**

EXAMPLE: Simplify $\dfrac{3}{x^2+5x+6}+\dfrac{2x}{x^2+4x+3}$.

Steps to Simplify:

STEP 1:

Factor the denominators.

$$\frac{3}{(x+2)(x+3)}+\frac{2x}{(x+1)(x+3)}$$

STEP 2:

Find the LCD.

The LCD is

__________________________________.

STEP 3:

Rewrite with equivalent fractions so that the fractions have the same name.

$$\frac{3}{(x+2)(x+3)}+\frac{2x}{(x+1)(x+3)}=$$

$$\frac{3(\_\_\_\_\_\_\_\_)}{\_\_\_\_\_\_\_\_\_\_}+\frac{2x(\_\_\_\_\_\_\_\_)}{\_\_\_\_\_\_\_\_\_\_}$$

STEP 4:

Simplify the fractions.

$$\frac{(\_\_\_\_\_\_\_\_)}{(x+1)(x+2)(x+3)}+\frac{(\_\_\_\_\_\_\_\_)}{(x+1)(x+2)(x+3)}=$$

$$\frac{\_\_\_\_\_\_\_\_\_\_}{(x+1)(x+2)(x+3)}$$

STEP 5:

If possible, factor the numerator and determine if the result can be simplified further.

$$\frac{2x^2+7x+3}{(x+1)(x+2)(x+3)}=\frac{(\_\_\_\_\_)(\_\_\_\_\_)}{(x+1)(x+2)(x+3)}$$

So, $\dfrac{3}{x^2+5x+6}+\dfrac{2x}{x^2+4x+3}=\dfrac{(\_\_\_\_\_)}{(x+1)(x+2)}$.

Note that we could have solved this problem by not factoring and using the trinomials as the common denominator. It will simplify to the same answer, but the terms could be complicated and harder to factor in Step 4.

• Adding and Subtracting Rational Expressions •

Simplify by factoring and then using the LCD.

1) $\frac{x-2}{x+1} - \frac{14x-21}{2x^2-x-3}$

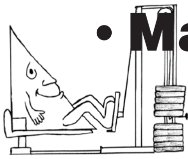
FACTORED: $\frac{x-2}{\_\_\_\_\_\_\_\_} - \frac{14x-21}{\_\_\_\_\_\_\_\_}$

LCD = __________________

____

2) $\frac{1}{x^2-3x-4} - \frac{2x}{x^2+8x+7}$

____



• Mastery Check: Combining Rational Expressions • ☆

Simplify.

1) $\dfrac{5x}{x^2 - 9} + \dfrac{4x}{x^2 + 5x + 6}$

____

2) $\dfrac{x}{x^2 + 4x + 3} - \dfrac{3}{x^2 - 4x - 5}$

____

Challenge:

3) $7 + \dfrac{x}{x^2 - 2x + 1} - \dfrac{1}{x^3 - 1}$

____

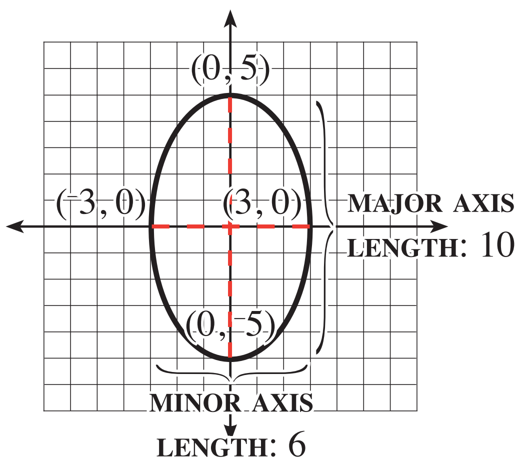
• Major and Minor Axes of an Ellipse • ☆

An **ELLIPSE** is an oval shape that has two axes of symmetry. The **MAJOR AXIS** is the longest diameter of the ellipse, and the endpoints are called the **VERTICES** of the ellipse. The **MINOR AXIS** is the smallest diameter of the ellipse, and the endpoints are called the **CO-VERTICES**.

> Recall that a diameter is a line that goes through the center and connects to two ends of the figure. Unlike a circle, the diameters of an ellipse are not all the same.

Direct Teaching

EXAMPLE: Determine the vertices, co-vertices, the length of the major axis, and the length of the minor axis of the ellipse.



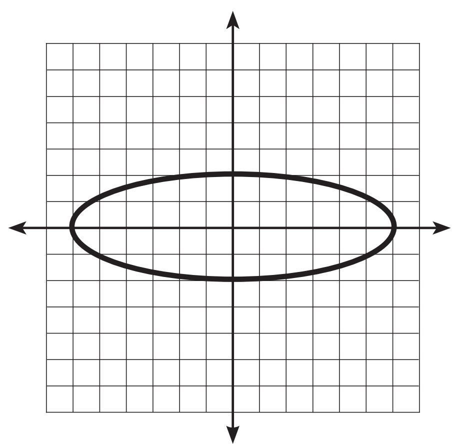
This ellipse is centered at the origin.

The longest diameter for this ellipse goes from the point (0, 5) to the point (0, $^-5$). This means that (0, 5) to (0, $^-5$) are the vertices; the major axis length is 10.

The shortest diameter for this ellipse goes from the point ($^-3$, 0) to the point (3, 0). This means that ($^-3$, 0) and (3, 0) are the co-vertices; the minor axis length is 6.

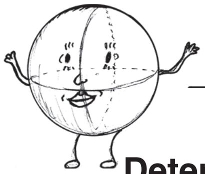
Try this: **Determine the vertices, co-vertices, the length of the major axis, and the length of the minor axis of the ellipse.**

1)



VERTICES: _________ and _________ **CO-VERTICES:** _________ and _________

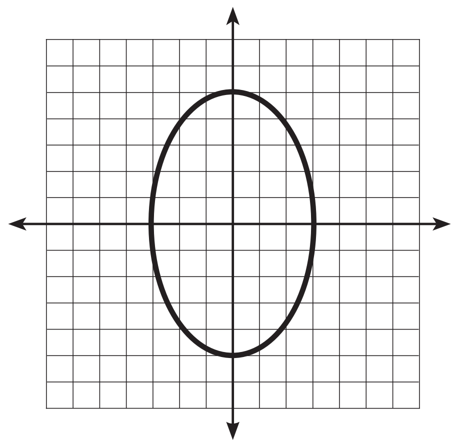
MAJOR AXIS LENGTH: ______________ **MINOR AXIS LENGTH:** ______________



• Major and Minor Axes of an Ellipse •

Determine the vertices, co-vertices, the length of the major axis, and the length of the minor axis of the ellipse.

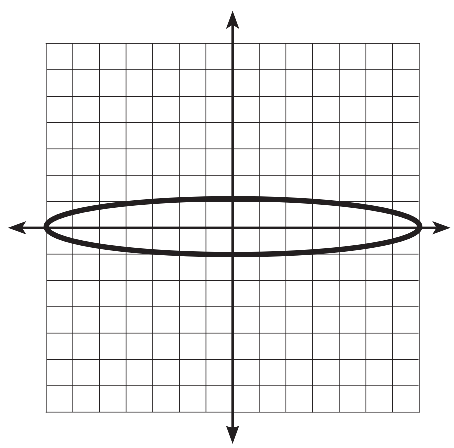
1)



VERTICES: __________ and __________ CO-VERTICES: __________ and __________

MAJOR AXIS LENGTH: _______________ MINOR AXIS LENGTH: _______________

2)



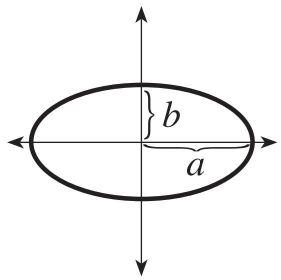
VERTICES: __________ and __________ CO-VERTICES: __________ and __________

MAJOR AXIS LENGTH: _______________ MINOR AXIS LENGTH: _______________

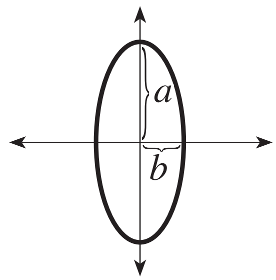
• Graphing an Ellipse Centered at the Origin • ☆

Direct Teaching

One way to view an ellipse is as a circle that is being compressed or stretched. Instead of being equal to the radius $\boldsymbol{r}$, the equation of an ellipse should be set equal to 1.



$$\frac{\boldsymbol{x}^2}{a^2} + \frac{\boldsymbol{y}^2}{b^2} = 1$$



$$\frac{\boldsymbol{y}^2}{a^2} + \frac{\boldsymbol{x}^2}{b^2} = 1$$

Notice that since a corresponds with the major axis, a will always be larger than b. So, if a is under $\boldsymbol{x}^2$, then the ellipse is longer on the $\boldsymbol{x}$–axis. Similarly if a is under $\boldsymbol{y}^2$, then the ellipse is longer on the $\boldsymbol{y}$–axis. If a and b are the same, then we have a circle.

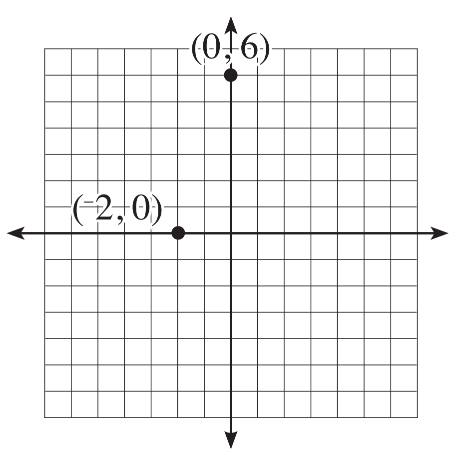
For an ellipse, $2a$ is the length of the major axis and $2b$ is the length of the minor axis. This means that the vertices are distance a from the center and the co-vertices are distance b from the center.

Try this:

Example: Graph the ellipse $\frac{\boldsymbol{y}^2}{36} + \frac{\boldsymbol{x}^2}{4} = 1$.

$a^2 = 36$. That means $a =$ ______. Since a is under $\boldsymbol{y}^2$, that means the ellipse is longer on the ________. So, the coordinates of the vertices are (0, 6) and (_____, _____).

$b^2 = 4$. That means $b =$ ______. Since b is under $\boldsymbol{x}^2$, that means the ellipse is smaller on the ________. So, the coordinates of the co-vertices are ($^-2$, 0) and (_____, _____).



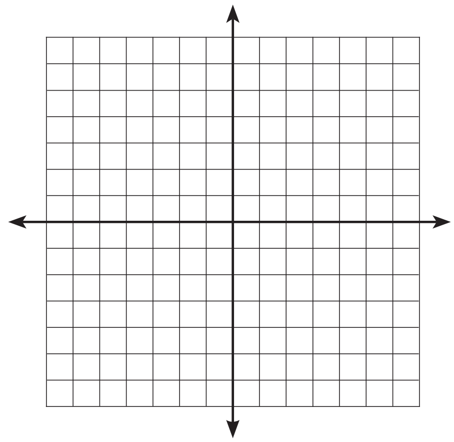
• Graphing an Ellipse Centered at the Origin • ☆

Determine the value of *a* and *b*. Then graph the ellipse. Label the vertices and co-vertices on the graph.

1) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

$a^2 = 25$, so $a =$ ___________

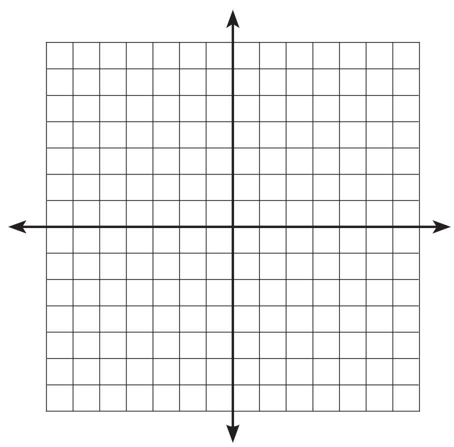
$b^2 = 9$, so $b =$ ___________



2) $\dfrac{y^2}{16} + \dfrac{x^2}{4} = 1$

$a =$ ___________

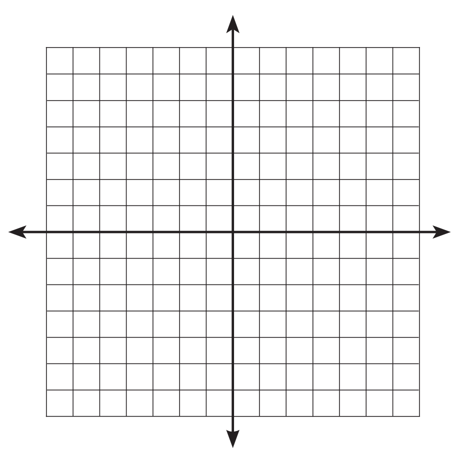
$b =$ ___________



3) $\dfrac{x^2}{49} + \dfrac{y^2}{1} = 1$

$a =$ ___________

$b =$ ___________



• Graphing an Ellipse Not Centered at the Origin •

The equation for the ellipse centered at (h, k) is given by one of the following equations.

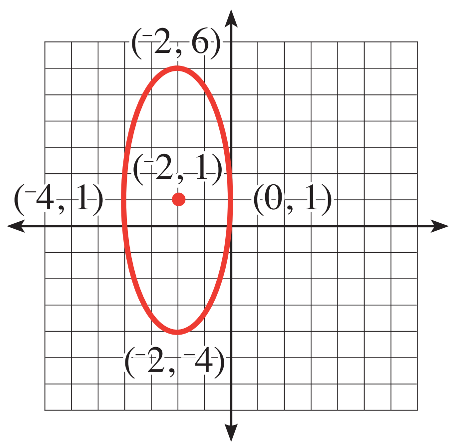
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

Example: Graph the ellipse $\frac{(y-1)^2}{25} + \frac{(x+2)^2}{4} = 1$.

Here, $h = {}^-2$ and $k = 1$. So, the center is at $({}^-2, 1)$.

Since a is under the y term, the major axis is parallel to the y–axis. $a = 5$, so the vertices are 5 units up and 5 units down from the center. That means the vertices are at $({}^-2, 6)$ and $({}^-2, {}^-4)$.

Since $b = 2$, the co-vertices are 2 units to the left and 2 units to the right from the center. That means the co-vertices are at $({}^-4, 1)$ and $(0, 1)$.



Try this: **Graph the ellipse.**

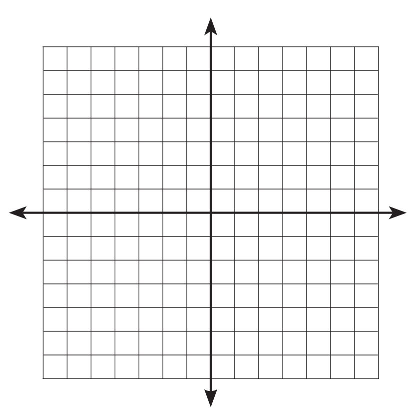
1) $\frac{(x+2)^2}{9} + \frac{(y-1)^2}{4} = 1$

$a =$ _____ $b =$ _____

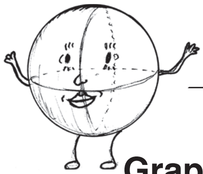
CENTER: ____________

VERTICES: __________ and __________

CO-VERTICES: __________ and __________



Direct Teaching



• Graphing an Ellipse •

Graph the ellipse. Determine the center, vertices, and co-vertices. Leave your answers in radical form.

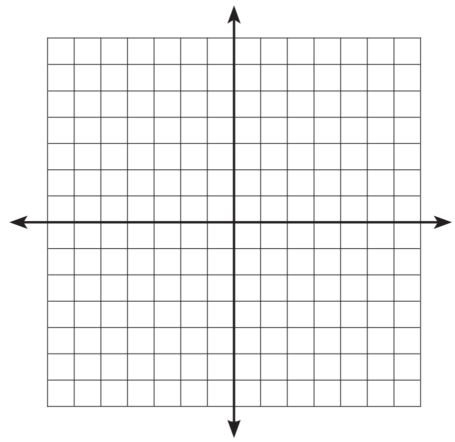
1) $\frac{(y-1)^2}{16} + \frac{(x-3)^2}{9} = 1$

$a =$ ______ $b =$ ______

CENTER: ______________

VERTICES: ____________ and ____________

CO-VERTICES: ____________ and ____________

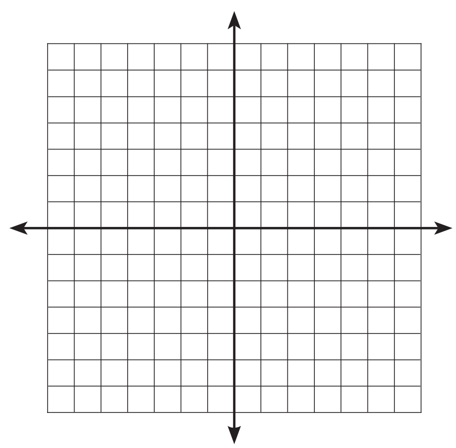


2) $\frac{(x+4)^2}{4} + (y+5)^2 = 1$

CENTER: ______________

VERTICES: ____________ and ____________

CO-VERTICES: ____________ and ____________



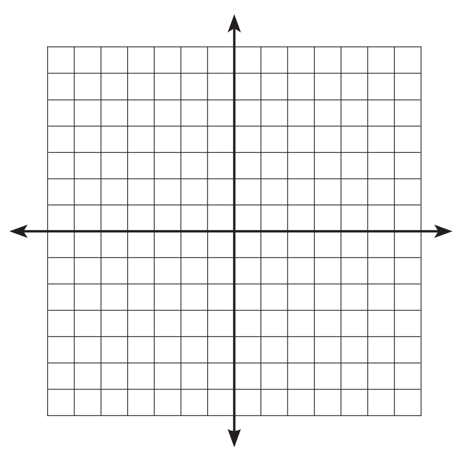
3) $\frac{x^2}{8} + \frac{y^2}{3} = 1$

Extending Knowledge

CENTER: ______________

VERTICES: ____________ and ____________

CO-VERTICES: ____________ and ____________



• Equation of an Ellipse: Completing the Square •

Sometimes we have to use completing the square multiple times to rewrite the equation of an ellipse into a form we can graph.

Example: Graph $x^2 + 4y^2 - 4x + 24y = {}^{-}24$.

First, we have to rewrite the equation so that all of the x terms are together and all of the y terms are together.

$$x^2 - 4x + 4y^2 + 24y = {}^{-}24$$

Now, let's complete the square for both the x terms and the y terms. Recall that to complete the square, we add $\left(\frac{b}{2}\right)^2$ to both sides of the equation. For the y terms, we need to factor out a 4 first before completing the square.

$$(x^2 - 4x + \_\_\_\_) + 4(y^2 + 6y + \_\_\_\_) = {}^{-}24$$

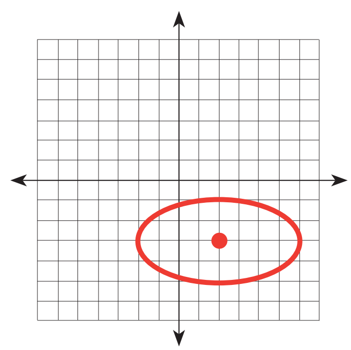
$$(x^2 - 4x + 4) + 4(y^2 + 6y + 9) = {}^{-}24 + 4 + 36$$

$$(x - 2)^2 + 4(y + 3)^2 = 16$$

In order to be equal to 1, we divide both sides by 16.

$$\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$$

Now that we have rewritten the equation of the ellipse, we can graph the ellipse. The center is at $(2, {}^{-}3)$, the major axis is 8 units long, and the minor axis is 4 units long.



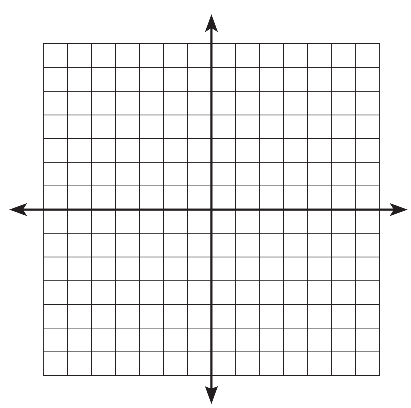
Try this: **Graph the ellipse. Determine the center and the length of the major and minor axes.**

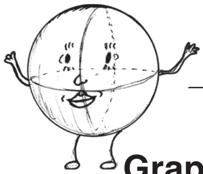
1) $y^2 - 2y + 4x^2 + 16x = {}^{-}1$

CENTER: _______________

LENGTH OF MAJOR AXIS: _______________

LENGTH OF MINOR AXIS: _______________

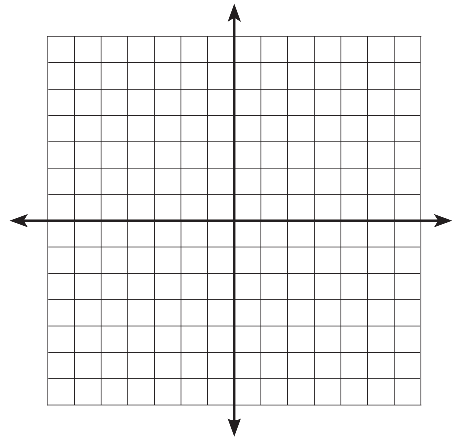




• Graphing an Ellipse •

Graph the ellipse. Determine the center and the length of the major and minor axes.

1) $4x^2 - 8x + 9y^2 + 54y = {}^{-}49$

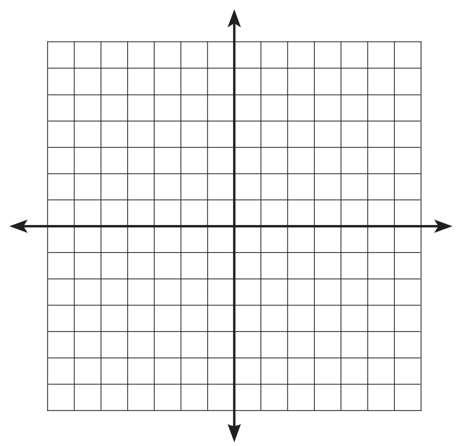


CENTER: ________________

LENGTH OF MAJOR AXIS: ________________

LENGTH OF MINOR AXIS: ________________

2) $\dfrac{x^2}{49} + \dfrac{(y+1)^2}{36} = 1$

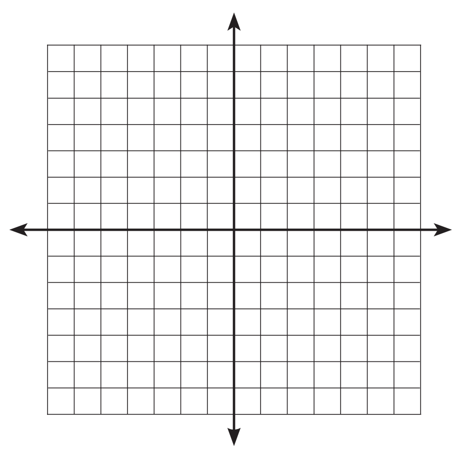


CENTER: ________________

LENGTH OF MAJOR AXIS: ________________

LENGTH OF MINOR AXIS: ________________

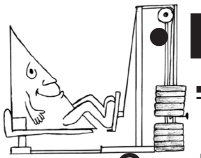
3) $y^2 + 49x^2 - 196x = {}^{-}147$



CENTER: ________________

LENGTH OF MAJOR AXIS: ________________

LENGTH OF MINOR AXIS: ________________



• Mastery Check: Graphing an Ellipse •

☆

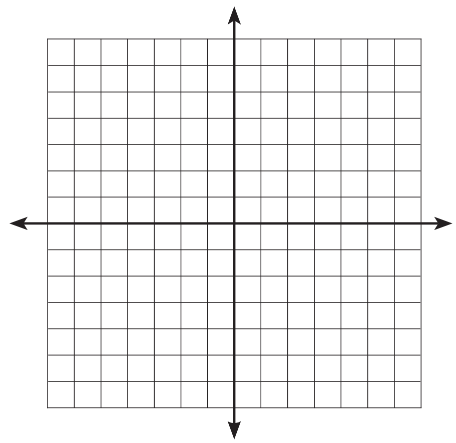
Graph the ellipse. Determine the center and the length of the major and minor axes.

1) $\frac{(y+1)^2}{36} + \frac{(x-2)^2}{9} = 1$

CENTER: ________________

LENGTH OF MAJOR AXIS: ________________

LENGTH OF MINOR AXIS: ________________

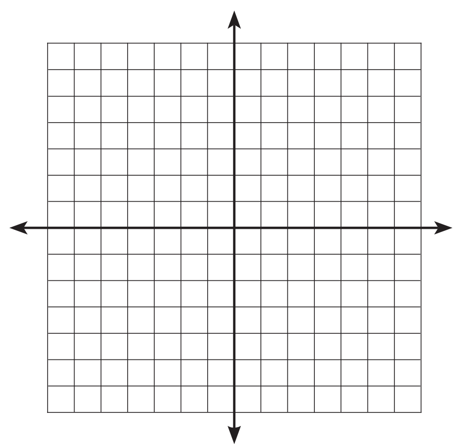


2) $x^2 + 4x + 4y^2 + 40y = {}^{-}88$

CENTER: ________________

LENGTH OF MAJOR AXIS: ________________

LENGTH OF MINOR AXIS: ________________



Challenge:

3) Write the equation of an ellipse that is centered at $(2, 3)$, has a vertex at $(2, 10)$, and has a minor axis length of 12.

____

• Rational Root Theorem: $\frac{p}{q}$ •

Direct Teaching

We can find the potential rational roots to any polynomial using the **RATIONAL ROOT THEOREM**.

Rational Root Theorem

Let p represent the factors of the constant term of a polynomial.

Let q represent the factors of the leading coefficient of a polynomial.

The list of possible rational roots of a polynomial are of the form $\pm\frac{p}{q}$.

We can use the rational root theorem to give us a list of potential roots to a polynomial. Let's see how to find all of the potential roots.

Try this:

EXAMPLE: Determine the list of possible rational roots of the polynomial $f(x) = \underset{q}{\underline{3}}x^3 + 4x^2 - 2x + \underset{p}{\underline{6}}$.

First, let's find p, which is the list of all factors of the constant term.

The constant term is 6. The factors of 6 are _______________ .

Now, let's find q, which is the list of all factors of the leading coefficient.

The leading coefficient is 3. The factors of 3 are _______________.

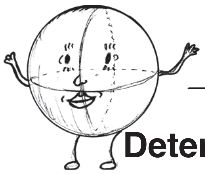
To find the list of possible rational roots to the polynomial, we find the value of $\frac{p}{q}$ for each p and each q. Fill in the missing values in the table below.

VALUES OF p AND q	$p = 1$ $q = 1$	$p = 2$ $q = 1$	$p = 3$ $q = 1$	$p = 6$ $q = 1$	$p = 1$ $q = 3$	$p = 2$ $q = 3$	$p = 3$ $q = 3$	$p = 6$ $q = 3$
$\pm\frac{p}{q}$	± 1	± 2			$\pm\frac{1}{3}$			

Note that sometimes numbers may appear in the list multiple times, but we only include them once in the final list.

So, the list of possible rational roots to the polynomial are:

$\pm 1, \pm 2,$ _____, _____, $\pm\frac{1}{3}$, and _____.



• Rational Root Theorem: $\frac{p}{q}$ •

Determine the list of possible rational roots of the polynomial.

1) $f(x) = x^3 + 4x^2 - 21x - 20$

FACTORS OF CONSTANT TERM (p): ____________________

FACTORS OF LEADING COEFFICIENT (q): ____________________

LIST OF POSSIBLE RATIONAL ROOTS: $\pm \frac{p}{q} =$ ____________________

2) $f(x) = 2x^4 + 3x^3 + 4x^2 - 5x - 6$

p: ____________________

q: ____________________

$\pm \frac{p}{q} =$ ____________________

3) $f(x) = x^2 + 40x - 36$

$\pm \frac{p}{q} =$ ____________________

4) $f(x) = 3x^3 - x^2 - 24$

$\pm \frac{p}{q} =$ ____________________

• Using the Rational Root Theorem •

Direct Teaching

Now that we know how to list all of the possible rational roots, we can use that list to find the rational roots of a polynomial.

Example: Determine the roots of the polynomial $f(x) = x^3 + 6x^2 - x - 30$.

First, let's find all of the possible rational roots.

VALUES OF p AND q	$p = 1$ $q = 1$	$p = 2$ $q = 1$	$p = 3$ $q = 1$	$p = 5$ $q = 1$	$p = 6$ $q = 1$	$p = 10$ $q = 1$	$p = 15$ $q = 1$	$p = 30$ $q = 1$
$\pm \frac{p}{q}$	± 1	± 2	± 3	± 5	± 6	± 10	± 15	± 30

Now, let's use synthetic division to determine which numbers in this list are rational roots of the polynomial. If we do synthetic division and do not have a remainder, then we found a root. Let's try $^{-}3$.

$^{-}3$	1	6	$^{-}1$	$^{-}30$
		$^{-}3$	$^{-}9$	30
	1	3	$^{-}10$	0

Since we got a 0 remainder, we know that $^{-}3$ is a root. This also means that $(x + 3)$ is a factor of $x^3 + 6x^2 - x - 30$. And now that we have reduced our polynomial to a quadratic, we can find the remaining roots by factoring or using the quadratic formula.

$$x^2 + 3x - 10 = (x - 2)(x + 5)$$

So, the roots of the polynomial $x^3 + 6x^2 - x - 30$ are $^{-}5$, $^{-}3$, and 2. This means that the factored form of $x^3 + 6x^2 - x - 30$ is $(x + 5)(x + 3)(x - 2)$.

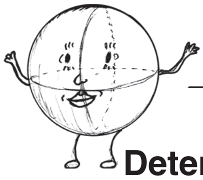
Note that both 2 and $^{-}5$ are in our list of possible rational roots as well. That means we could have done synthetic division with those first and still have gotten the same answer.

Try this: **Determine a root of the polynomial and write it as a factor.**

1) $f(x) = x^3 + 2x^2 - 21x + 18$ $\pm \frac{p}{q} =$ ______________________________

____

ROOT: __________________ FACTOR: __________________



• Using the Rational Root Theorem •

Determine a root of the polynomial and write it as a factor.

1) $f(x) = x^3 - 8x^2 + 11x + 20$ $\pm \frac{p}{q} =$ ______________________

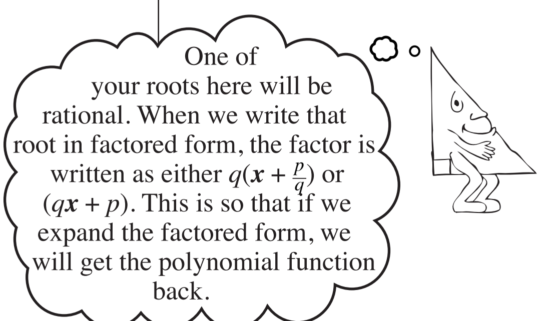
ROOT: ______________ FACTOR: ______________

2) $f(x) = 2x^3 - x^2 - 13x - 6$

ROOT: ______________ FACTOR: ______________

List the possible roots. Then determine all roots of the polynomial. Write the polynomial in factored form.

3) $f(x) = 3x^3 - 17x^2 - 8x + 12$ $\pm \frac{p}{q} =$ ______________________



ROOTS: ______________ FACTORED FORM: ______________

• Factor Theorem •

Direct Teaching

Sometimes we can get a long list of possible rational roots, and it may take a long time to solve if you do not choose a correct rational root in the first few attempts. There is a strategy to help determine which roots to try.

Factor Theorem

Given a polynomial $f(x)$, if there is a value of c such that $f(c) = 0$, then $x - c$ is a factor of $f(x)$.

What this means is if we substitute a possible rational root into the polynomial and get a value of 0, then that value is a rational root of the polynomial.

Example: Determine a root of the polynomial $f(x) = x^3 + 8x^2 + 11x - 20$.

First, let's find all of the possible rational roots.

VALUES OF p AND q	$p = 1$ $q = 1$	$p = 2$ $q = 1$	$p = 4$ $q = 1$	$p = 5$ $q = 1$	$p = 10$ $q = 1$	$p = 20$ $q = 1$
$\pm \frac{p}{q}$	± 1	± 2	± 4	± 5	± 10	± 20

There are 12 possible values to try. However, we can get a better idea of which one(s) to try if we substitute a value for x and the polynomial simplifies to 0. A good starting point is smaller numbers since the calculations can usually be done mentally. Let's try $x = 1$.

$$f(1) = (1)^3 + 8(1)^2 + 11(1) - 20 = 1 + 8 + 11 - 20 = 0$$

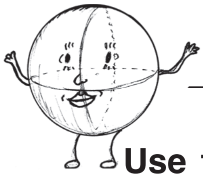
Since the value is 0, that means 1 is a root to the polynomial. Note that we can use synthetic division to find all of the roots.

***Try this*: Determine a root of the polynomial and write it as a factor.**

1) $f(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$ $\qquad \pm \frac{p}{q} =$ ____________________

$f(\_\_) =$ ____

ROOT: ______________ FACTOR: ______________



• Rational Root Theorem •

Use the Rational Root Theorem to determine the roots to the polynomial. Write the polynomial in factored form.

1) $f(\boldsymbol{x}) = 3\boldsymbol{x}^3 + \boldsymbol{x}^2 - 12\boldsymbol{x} - 4$ $\qquad \pm\frac{p}{q} =$ ______________________________

$f(\_\_) =$ ____

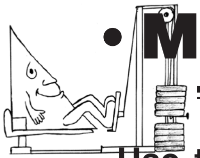
ROOTS: __________________ FACTORED FORM: __________________

2) $f(\boldsymbol{x}) = 2\boldsymbol{x}^4 + 9\boldsymbol{x}^3 - 9\boldsymbol{x}^2 - 46\boldsymbol{x} + 24$

ROOTS: __________________ FACTORED FORM: __________________

3) $f(\boldsymbol{x}) = \boldsymbol{x}^4 - \boldsymbol{x}^3 - 17\boldsymbol{x}^2 + 21\boldsymbol{x} + 36$

ROOTS: __________________ FACTORED FORM: __________________



• **Mastery Check:** Rational Root Theorem • ☆

Use the Rational Root Theorem to determine the roots to the polynomial. Write the polynomial in factored form.

1) $f(x) = 2x^3 + 5x^2 - 28x - 15$

ROOTS: ____________________ FACTORED FORM: ____________________

2) $f(x) = x^4 + 4x^3 - 27x^2 - 34x + 56$

ROOTS: ____________________ FACTORED FORM: ____________________

Challenge:

3) Determine the x–intercepts of the function $f(x) = x^3 + 4x^2 + 2x - 3$.

X–INTERCEPTS: ____

• Powers of i •

Recall that the imaginary number $i = \sqrt{-1}$. Let's take a look at what happens when i is raised to various powers.

Try this:

Let's try finding i^2 first.

$$i^2 = i \bullet i = \sqrt{-1} \bullet \sqrt{-1} = \underline{\hspace{3cm}}$$

So, $i^2 = \underline{\hspace{3cm}}$

Now, let's try finding all powers of i up to 8.

Point Out a Pattern

$$i^3 = i^2 \bullet i = \underline{\;-1\;} \bullet \underline{\;i\;} = \underline{\hspace{3cm}}$$

$$i^4 = i^2 \bullet i^2 = \underline{\hspace{2cm}} \bullet \underline{\hspace{2cm}} = \underline{\hspace{3cm}}$$

$$i^5 = i^4 \bullet i = \underline{\hspace{2cm}} \bullet \underline{\hspace{2cm}} = \underline{\hspace{3cm}}$$

$$i^6 = i^4 \bullet i^2 = \underline{\hspace{2cm}} \bullet \underline{\hspace{2cm}} = \underline{\hspace{3cm}}$$

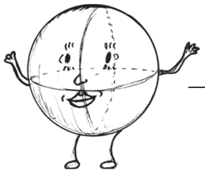
$$i^7 = i^4 \bullet i^3 = \underline{\hspace{2cm}} \bullet \underline{\hspace{2cm}} = \underline{\hspace{3cm}}$$

$$i^8 = i^4 \bullet i^4 = \underline{\hspace{2cm}} \bullet \underline{\hspace{2cm}} = \underline{\hspace{3cm}}$$

What do you notice about the powers of i?

____

____



• Powers of i •

Direct Teaching

Since $i^4 = 1$, we can simplify i raised to any power greater than 4 by taking out all i^4. We can do this by dividing the exponent by 4 and looking at the remainder. Alternatively, we can try to find the closest multiple of 4 to the exponent of i. Let's look at an example using both methods.

Example: Simplify i^{91}.

Writing out products of i^4 would be long and tedious. So, let's divide 91 by 4 and look at the remainder.

$$91 \div 4 = 22 \text{ R } 3$$

This means we will have 22 i^4 terms, and the remainder of 3 means the last term is i^3. This means that i^{91} simplifies to i^3, which is ^-i.

Alternatively, i raised to any multiple of 4 will be 1. That means we can find the highest multiple of 4 that is less than 91 and use that to simplify i^{91}. We can rewrite i^{91} as:

$$i^{91} = i^{88} \bullet i^3 = (i^4)^{22} \bullet i^3 = 1 \bullet i^3 = {}^-i$$

So, $i^{91} = {}^-i$.

Try these: **Simplify.**

1) i^{58}

$58 \div 4 =$ ______ R ______

____

2) i^{95}

$i^{95} = i^{\square} \bullet i^{\square} = (i^4)^{\square} \bullet i^{\square}$

____

3) i^{124}

____

4) i^{49}

____

5) i^{55}

____

6) $i^{1,001}$

____

• Adding and Subtracting Complex Numbers •

A COMPLEX NUMBER (z) is of the form $a + bi$, where a and b are real numbers. a is the real part and bi is the imaginary part. If $a = 0$, then you have a pure imaginary number. If $b = 0$, then you have a real number.


The Law of SAMEness

When we add or subtract complex numbers, we combine the real parts of each number, and then we combine the imaginary parts of each number separately. Let's look at an example.

EXAMPLE: Subtract $(1 + 3i)$ from $(3 + 2i)$.

The two complex numbers are $3 + 2i$ and $1 + 3i$. When we subtract the real parts, we have $3 - 1 = 2$. When we subtract the imaginary parts, we have $2i - 3i = {}^{-}i$. So,

$$(3 + 2i) - (1 + 3i) = 3 + 2i - 1 - 3i = 3 - 1 + 2i - 3i = 2 - i$$

Notice that this process is the same as combining like terms for real numbers.

***Try these*: Simplify.**

1) $(7 - 5i) + (6 + 8i)$

____

2) $(3 + 2i) - (5 + 9i)$

____

3) ${}^{-}6 + 8i - (1 - 10i)$

____

4) $8 + 12i + 1 - 4i - 2 + i$

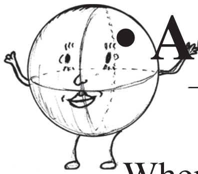
____

5) $(15 - 6i) - (9 - 18i)$

____

6) ${}^{-}13 - 5i + 8 - 9i$

____



• Adding and Subtracting Complex Numbers • ☆

When i is raised to a power greater than 1, we simplify the result so that the expression is of the form $a + bi$.

Example: Simplify $^{-}8i + 10 + 3i^2$.

$$^{-}8i + 10 + 3i^2 = {}^{-}8i + 10 + 3(^{-}1) = {}^{-}8i + 7 = 7 - 8i$$

A complex number is always written in the form $a + bi$. Make sure when simplifying that the simplified form is written as $a + bi$.

***Try these*: Simplify.**

1) $12 + 4i^2 + 6 - 4i + i^2$

____

2) $^{-}18 + i + 10 + 19i$

____

3) $2 - 8i^2 + 1 - 9i^2$

____

4) $i + i^2 + i^3 + i^4$

____

5) $19 - 5i^2 - 8i^3$

____

6) $13 - (9i^5 - 5 + 10i^2)$

____

• Multiplying Complex Numbers •

The process of multiplying complex numbers is the same process as multiplying binomials.

Example: Simplify $(3 - 4i)(2 + 5i)$.

By distributing the terms, the expression simplifies to the following:

$$(3 - 4i)(2 + 5i) = 6 + 15i - 8i - 20i^2 = 6 + 7i - 20i^2$$

$$6 + 7i - 20i^2 = 6 + 7i - 20(^-1) = 6 + 7i + 20 = 26 + 7i$$

So, $(3 - 4i)(2 + 5i)$ simplifies to $26 + 7i$.

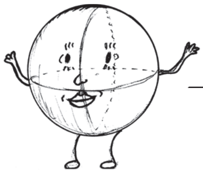
Try these: **Simplify.**

1) $(4 - 11i)(3 - 2i)$ ____________________

2) $(1 + 2i)(3 + 4i)$ ____________________

3) $(5 + 6i)(4 - 10i)$ ____________________

4) $(6 - 8i)(6 + 8i)$ ____________________



• Multiplying Complex Numbers •

Simplify.

1) $(6 + 5i)(10 - 3i)$

____

2) $(15 - i)(4 - 3i)$

____

3) $(10 + 6i)(5 - 9i)$

____

4) $(10 - 7i)(10 + 7i)$

____

5) $(3 + 8i)(1 - 10i^2)$

____

6) $(5 - 3i^2)(2 + 10i^2)$

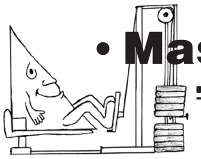
____

7) $(11 + i^3)(2 - 7i^2)$

____

8) $(4 - 3i^2)(1 + 5i^3)(1 + 2i^4)$

____



• Mastery Check: Adding and Multiplying Complex Numbers •

Simplify.

1) i^{35}

2) i^{102}

3) $2 + 4i - (7 + i)$

4) $10 - 9i + 4 - 2i + 6$

5) $(3 + 6i)(4 - 7i)$

6) $(8 - 3i)(11 - 4i)$

Challenge:

7) $\dfrac{8 + i - 3 + 14i}{(2 - i)(2 + i)}$

8) $(\boldsymbol{x} - (1 - 2i))(\boldsymbol{x} - (1 + 2i))$

• Exponential and Logarithmic Form •

Direct Teaching

An equation in the following form is considered to be exponential: $5^3 = 125$. Exponential equations can be written in LOGARITHMIC form. A logarithm, also called a log, represents the number of times we have to multiply a base to get a particular value. Here, we have to multiply the base 5, three times to get 125. Let's look at an equation written in exponential form and convert it to LOGARITHMIC form.

EXAMPLE: Write $5^3 = 125$ in logarithmic form.

Here, the base of the exponent is 5, and the exponent it is raised to is 3. This equals 125. Moving the terms into logarithmic form, this will be written:

$$\log_5 125 = 3$$

So, the log–base–5 of 125 equals 3. Notice that logarithms are exponents.

Try these: **Write the exponential equation in logarithmic form.**

1) $3^4 = 81$

BASE: __________

EXPONENT: ________

$\log_{\_\_\_} 81 = \_\_\_\_\_$ (BASE, EXPONENT)

2) $2^x = 64$

BASE: __________

EXPONENT: ________

$\log_{\_\_\_} 64 = \_\_\_\_\_$ (BASE, EXPONENT)

3) $5^4 = 625$

BASE: __________

EXPONENT: ________

$\log_{\_\_\_} 625 = \_\_\_\_\_$ (BASE, EXPONENT)

4) $4^3 = 64$

BASE: __________

EXPONENT: ________

$\log_{\_\_\_} 64 = \_\_\_\_\_$ (BASE, EXPONENT)

• Exponential to Logarithmic Form •

Write the exponential equation in logarithmic form.

1) $7^2 = 49$

BASE: ____________

EXPONENT: ________

$\log_{\underline{\text{BASE}}} 49 = \underline{\qquad}$ (EXPONENT)

2) $x^2 = 100$

BASE: ____________

EXPONENT: ________

____

3) $4^0 = 1$

____

4) $81^{\frac{1}{4}} = 3$

____

5) $4^x = 16$

____

6) $6^3 = 216$

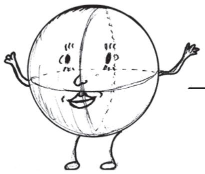
____

7) $7^{-2} = \frac{1}{49}$

____

8) $4^x = y$

____



• Logarithmic to Exponential Form •

Example: Write $\log_3 81 = 4$ in exponential form.

Here, the log–base–3 of 81 equals 4. We can move the terms into exponential form.

$$3^4 = 81$$

Try these: **Write the logarithmic equation in exponential form.**

1) $\log_2 8 = 3$

____

2) $\log_4 256 = 4$

____

3) $\log_2 64 = y$

____

4) $\log_3 3 = 1$

____

5) $\log_{\frac{1}{9}} 81 = {}^{-}2$

____

6) $\log_{12} 144 = 2$

____

7) $\log_{\frac{1}{3}} \frac{1}{9} = 2$

____

8) $\log_{36} 6 = \frac{1}{2}$

____

9) $\log_2 512 = 10$. Explain why the logarithmic equation is not valid.

____

____

• Logarithmic Form •

Direct Teaching

The base of a logarithm must be positive, and cannot equal 1. Let's look at a couple of examples.

Example 1: Write $\log_1 5 = x$ in exponential form. Is the original logarithm valid?

This is written $1^x = 5$. This asks, "1 raised to what power x equals 5?" There is not a power we could raise 1 to, and get 5 as an output. This is not a valid logarithm.

Example 2: Write $\log_{-3} 7 = x$ in exponential form. Is the original logarithm valid?

This is written $(^{-}3)^x = 7$. This asks, "$^{-}3$ raised to what power x equals 7?" There is not a power we could raise $^{-}3$ to, and get 7 as an output. This is not a valid logarithm.

Try these: **Write the logarithmic equation as an exponential equation. Is it a valid logarithm?**

1) $\log_{\frac{1}{5}} 125 = {}^{-}3$

____

2) $\log_0 3 = x$

____

3) $\log_2 {}^{-}5 = x$

____

4) $\log_1 2 = x$

____

5) $\log_{-5} x = y$

____

• Evaluating Logarithms •

Direct Teaching

Example: Evaluate $\log_5 625$.

Let's assume $\log_5 625$ equals $\boldsymbol{x}$, and write this in exponential form.

$$5^x = 625$$

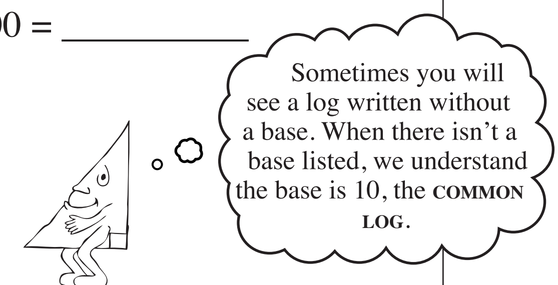
Notice that 625 happens to be a power of 5. If we can get the base of a logarithm to match the base of the independent variable, in this case 5, then the exponent this is raised to is our solution.

$$5^x = 5^4$$
$$\boldsymbol{x} = 4$$

***Try these*:** **Evaluate, if possible. Write "DNE" if it cannot be evaluated.**

1) $\log_5 25 =$ ___________

2) $\log 1{,}000 =$ ___________



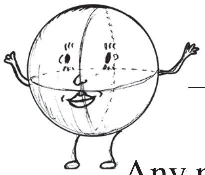
3) $\log_7 343 =$ ___________

4) $\log_2(^-2) =$ ___________

5) $\log_{\frac{1}{4}} \dfrac{1}{256} =$ ___________

6) $\log_{-2} 7 =$ ___________

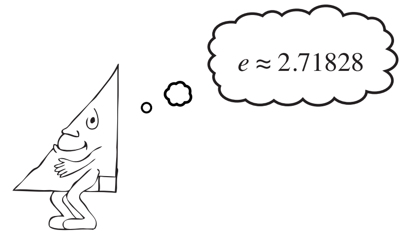
7) Why do you think $\log_{10}$ is called the common log? The number system that we use has a base of ___________.



• Natural Logarithm •

Any positive number not equal to 1 can be the base of a logarithm. When the base of a logarithm is e, it is called the **NATURAL LOG**, and it is abbreviated ln.

$$\log_e \boldsymbol{y} = \ln \boldsymbol{y}$$



EXAMPLE: Write $e^5 = \boldsymbol{y}$ in logarithmic form.

$$\log_e \boldsymbol{y} = 5$$

$\log_e$ is the natural logarithm so we will take one additional step to make this a natural logarithm. Notice, we no longer include the subscript e when writing natural logarithms.

$$\ln \boldsymbol{y} = 5$$

Try these: **Write the exponential equation in logarithmic form.**

1) $e^5 = \boldsymbol{x}$

____

2) $4^{-3} = \frac{1}{64}$

____

3) $2^5 = 32$

____

4) $e^{22} = \boldsymbol{x}$

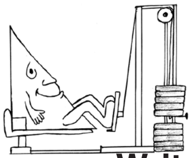
____

5) $2^x = 7$

____

6) $e^x = \boldsymbol{y}$

____



• **Mastery Check:** Logarithms •

Write the exponential equation in logarithmic form.

1) $2^x = 5$ ______________________________

2) $b^x = c$ ______________________________

Evaluate, if possible. Write "DNE" if it cannot be evaluated.

3) $\log_6 216 =$ ___________

4) $\log_3 81 =$ ___________

5) $\log_5(^-25) =$ ___________

6) $\log_{\frac{1}{8}} 64 =$ ___________

Challenge: **Evaluate, if possible. Write "DNE" if it cannot be evaluated.**

7) $\log_{16} 8$

____

• Quadratic Modeling •

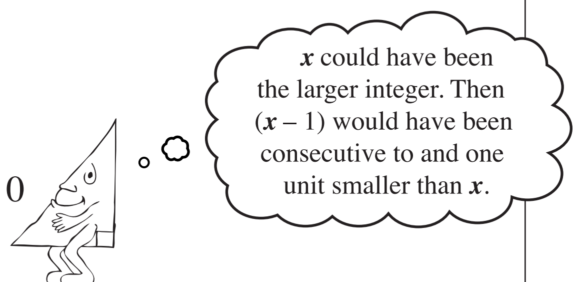
Direct Teaching

There are many situations that can be modeled using a quadratic function. We need to look at what we are given and use these facts in order to generate a model. Let's look at one such scenario.

Example: Two consecutive negative integers have a product of 156. What are their values?

Integers that are consecutive will be a single unit apart. If the smallest integer has a value of $\boldsymbol{x}$, an integer that is consecutive and one unit larger will have a value of $(\boldsymbol{x} + 1)$. Their product equals 156, so let's set up a quadratic model.

$$(x)(x + 1) = 156$$
$$x^2 + x = 156$$
$$x^2 + x - 156 = 0$$
$$(x + 13)(x - 12) = 0$$
$$x = 12, {}^{-}13$$



We are only interested in the negative integers, so $\boldsymbol{x} = {}^{-}13$. Plugging this back into the expression $(\boldsymbol{x} + 1)$ gives us all of the integers we are interested in.

$$x + 1 = {}^{-}13 + 1 = {}^{-}12$$

The two numbers are ${}^{-}12$ and ${}^{-}13$.

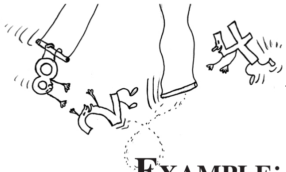
Try this:

1) Two consecutive negative integers have a product of 306. What are their values?

Expressions: x, ______________________

Quadratic Model: ______________________

____



• Quadratic Modeling •

Direct Teaching

Example: Two integers have a sum of 16. What is their maximum product?

Here we have multiple conditions that we have to consider, a sum and a product. Since we have two integers, one can be called $\boldsymbol{x}$ and the other $\boldsymbol{y}$. Their sum will then be $\boldsymbol{x} + \boldsymbol{y}$, and their product, $\boldsymbol{xy}$.

$$\boldsymbol{x} + \boldsymbol{y} = 16$$
$$\boldsymbol{xy} = ?$$

We can isolate one of the variables from the sum, and substitute it into the formula for the product.

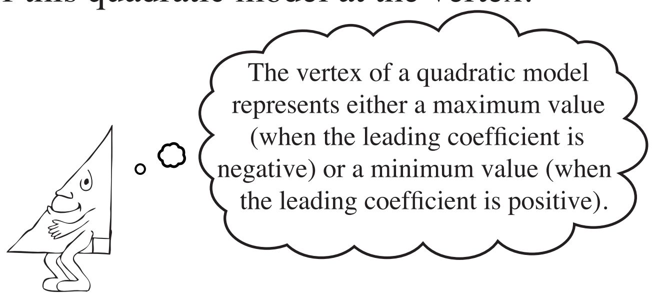
$$\boldsymbol{y} = {}^-\boldsymbol{x} + 16$$
$$\boldsymbol{xy} = \boldsymbol{x}({}^-\boldsymbol{x} + 16) = \boldsymbol{P}(\boldsymbol{x})$$
$${}^-\boldsymbol{x}^2 + 16\boldsymbol{x} = \boldsymbol{P}(\boldsymbol{x})$$

We can find the *maximum* value of this quadratic model at the vertex.

$$\boldsymbol{x} = \frac{{}^-b}{2a} = \frac{{}^-16}{2({}^-1)} = 8$$

Then solving for $\boldsymbol{y}$:

$$\boldsymbol{y} = {}^-\boldsymbol{x} + 16 = {}^-8 + 16 = 8$$

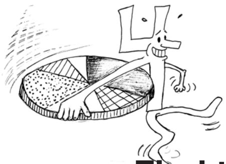


So, the integers that maximize the product are both 8, and their product is 64.

Try this: **Find the values of the integers described in the exercise.**

1) Two integers have a sum of 24. What is their maximum product?

____



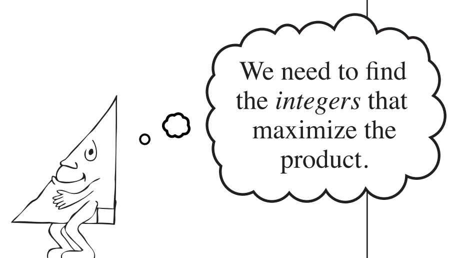
• Quadratic Modeling •

Find the values of the integers described in the exercise.

1) Two integers have a sum of 22. What is their maximum product?

____

2) Two integers have a sum of 15. What is their maximum product?



____

3) Two numbers have a sum of $17\frac{1}{2}$. What is their maximum product?

____

Direct Teaching

Let's look at a model involving a real world scenario with a fixed value.

Try this:

Example: OJ purchased 200 feet of fence to put a rectangular barrier around his orange grove. What are the dimensions and area of the largest possible rectangular grove OJ can fence?

We are being asked to find the dimensions and area of the largest possible rectangular grove that can be fenced. Let's start by writing down the formula for the area of a rectangle in terms of its length(***l***) and width(***w***).

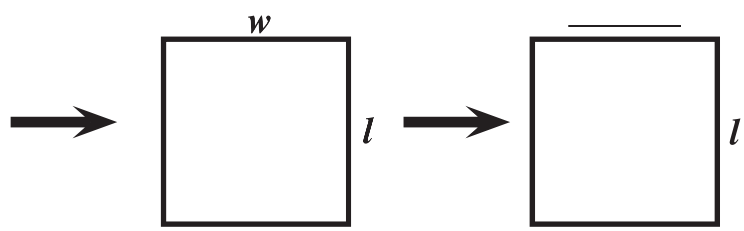
$$A = \underline{\ l\ } \bullet \underline{\quad\quad}$$

Next, we know the barrier is rectangular and has a fixed value of 200 feet. This allows us to write a formula for the perimeter. We then need to solve this equation for one of the variables, either ***l*** or ***w***. Let's solve for ***w***.

$P = \underline{\quad\quad} l + \underline{\quad\quad} w$

$200 = \underline{\quad\quad} l + \underline{\quad\quad} w$

$w = \underline{\quad\quad\quad\quad\quad}$



We now have an expression for the value of ***w*** in terms of ***l***. Let's now go back to the formula for the area and substitute in the expression we determined for ***w***. This will give us an equation which we can optimize, or find the maximum of.

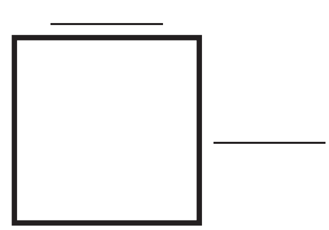
$$A = l(\underline{\quad\quad\quad\quad\quad})$$

$$A = \underline{\quad\quad\quad\quad\quad}$$

This is a quadratic model for the area in terms of the length. We can now find the *maximum* length from the vertex and solve for the maximum width that corresponds to this value.

$l = \frac{-b}{2a} = \frac{\underline{\quad\quad}}{2(\quad)} = \underline{\quad\quad}$ feet

$w = \underline{\quad\quad\quad\quad\quad} = \underline{\quad\quad}$ feet



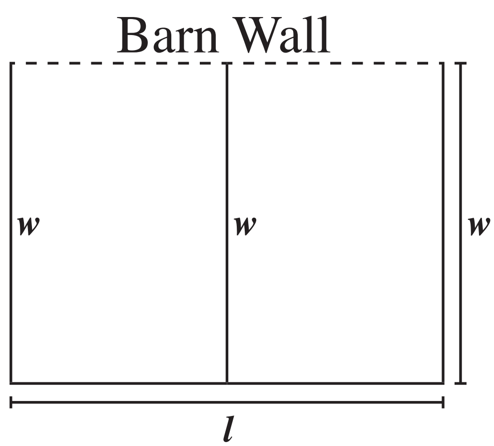
So, the rectangle with the maximum area turns out to have dimensions of ____________ × ____________, and an area of __________ square feet.

• Determining the Model – Fixed Perimeter •

Direct Teaching

Let's look at another fixed perimeter problem with additional conditions.

Example: OJ decided to try producing other fruits as well. In order to test two varieties of apples he uses 240 feet of fencing to create a rectangular orchard with two sections that evenly split the orchard into halves. One side is to be bordered by the barn wall. What is the area of the largest possible fenced orchard?



Steps to Solve:

Step 1: Write the formula for the area.	$A =$ ______________
Step 2: Determine a formula for the total perimeter from the scenario. Remember, one side is the barn wall.	$P =$ __________ $= 240$
Step 3: Solve the perimeter formula for either ***l*** or ***w***.	
Step 4: Substitute back into the area formula so that the area formula is in terms of a single variable, either ***l*** or ***w***.	
Step 5: Solve for the length or width that maximizes the area. Then solve for the other unknown value to find the area.	____________________________

• Quadratic Modeling •

Solve.

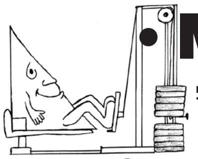
1) OJ keeps his tools in a rectangular shed which has a width 3 times that of the length. The area of the section is 192 square feet. What are the dimensions of the shed?

____

Draw a Picture

2) After a while OJ has determined what grows best in his area and begins developing a large rectangular farm to grow ten varieties of fruits and vegetables. He starts with 2,400 feet of fencing and sets it up as two rows with five different plants that each needs their own section of equal area. What are the dimensions of the largest possible fenced farm and its area?

DIMENSIONS: ______________________ **AREA**: ______________________



• Mastery Check: Quadratic Modeling •

Solve.

1) A farmer has 60 feet of fence to make a rectangular enclosure where one side of the enclosure is part of the barn wall. What is the maximum area of the enclosure?

____

2) Two consecutive even positive integers have a product of 728. What are their values?

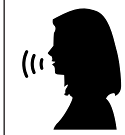
____

Challenge:

3) OJ plants a flower garden for one of his favorite customers, Magda. The garden will be 8 feet wide and 10 feet long. After planting the garden, OJ kept stepping on the flowers. In order to avoid this and to appreciate the flowers properly, he decides to put a walkway around the garden that is *uniformly wide*. The total area covered by the garden and walkway is now 288 square feet. How wide is the walkway?

____

• Factorials •


Direct Teaching

The **FACTORIAL** of a positive integer $\boldsymbol{n}$ is $\boldsymbol{n}$ multiplied by all positive integers less than $\boldsymbol{n}$. It is denoted by $\boldsymbol{n}!$ and can be read aloud as "n factorial."

$$1! = 1,\ 2! = 2 \times 1,\ 3! = 3 \times 2 \times 1, \ldots, \boldsymbol{n}! = \boldsymbol{n} \times (\boldsymbol{n} - 1) \times (\boldsymbol{n} - 2) \times \ldots \times 3 \times 2 \times 1$$

EXAMPLE 1: Calculate 5!.

The factorial of 5 is 5 multiplied by all positive integers less than 5. So,

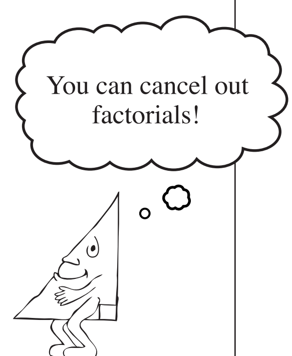
$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

We can use factorials in any math expression as it is shorthand for writing this repeated multiplication.

EXAMPLE 2: Simplify: $\frac{5!}{3!}$

Let's expand the numerator and denominator.

$$\frac{5!}{3!} = \frac{5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{3} \times \cancel{2} \times \cancel{1}} = 20$$



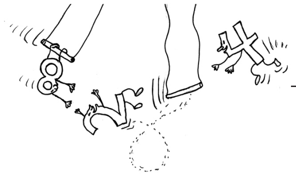
However, notice that $3 \times 2 \times 1$ is the same as 3!. So,

$$\frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3!} = \frac{5 \times 4 \times \cancel{3!}}{\cancel{3!}} = 20.$$

Try these:

1) $4! =$ ________ **2)** $6! =$ ________

3) $\frac{9!}{8!} =$ ________ **4)** $\frac{7!}{5!} =$ ________



• Factorials •

1) $3! =$ ________

2) $1! =$ ________

3) $\frac{7!}{4!} =$ ________

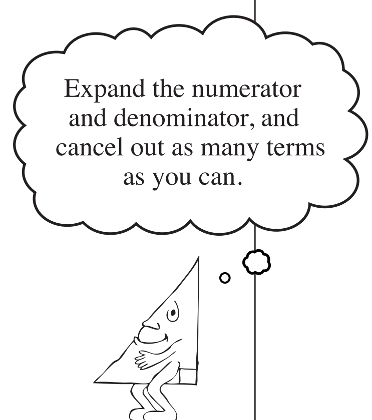
4) $\frac{6!}{5!} =$ ________

5) $\frac{8!}{6!} =$ ________

6) $\frac{12!}{9!} =$ ________

7) $\frac{9!}{5!\,4!} =$ ________

8) $\frac{11!}{7!\,4!} =$ ________



9) $\frac{14!}{4!\,11!} =$ ________

10) $\frac{10!}{5!\,5!} =$ ________

11) $\frac{13!}{8!\,5!} =$ ________

12) $\frac{12!}{9!\,3!} =$ ________

• Permutations •

Direct Teaching

A PERMUTATION of a set of objects is any way we can arrange them where their order matters.

EXAMPLE: Albert (*A*), Beth (*B*), and Carrie (*C*) are in a race. How many different ways could they end up in first, second, and third place?

Here, the order of the racers matters as ***ABC*** (Albert placing first, Beth second, and Carrie third) is a different result than ***ACB*** (Albert placing first, Carrie second, and Beth third). Each arrangement of the racers is a PERMUTATION of the race results.

Below are all the possible ways Albert (*A*), Beth (*B*), and Carrie (*C*) can end up in first, second, and third place. Notice there are 6 unique finishes.

ABC	***ACB***	***BAC***	***BCA***	***CAB***	***CBA***

We can also use the **FUNDAMENTAL COUNTING PRINCIPLE** to find out the total number of possibilities. Since either Albert, Beth, or Carrie could finish first, there are 3 possible first place finishers. After someone has finished first, there are 2 possible second place finishers. And finally, after someone has finished second, there is only 1 option for third place.

So, the number of PERMUTATIONS of the race results is $3 \times 2 \times 1 = 6$ or $3!$.

> The total number of PERMUTATIONS of ***n*** objects is ***n***!.

Try this:

1) Cindy has 5 figurines to put on her shelf. She decides to put them all in a row. How many unique ways are there to arrange the figurines?

____

• Permutations •

Use a Calculator

1) In a race between 7 people, how many unique ways could the racers finish?

____

2) Ryan has 6 books to read. How many unique ways could he read the books in order?

____

3) In a deck of 10 unique cards, how many unique ways could the cards be shuffled?

____

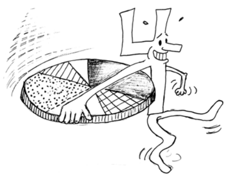
4) Jake is making a playlist from 9 songs. How many unique ways could he order the playlist?

____

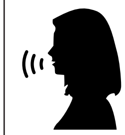
Extending Knowledge

5) Sally has 0 trophies. How many unique ways could she arrange her trophies on her shelf?

____



• Permutations •



Direct Teaching

Example: In a race between 5 people, how many different ways could first, second, and third place end up?

Again, the order of the racers matters, so we are looking for the number of permutations of the top three finishes. There are 5 racers that could end in first, which leaves 4 racers who could end in second, and 3 racers that could end in third. Using The Fundamental Counting Principle, we find that there are $5 \times 4 \times 3 = 60$ permutations of the top three finishes.

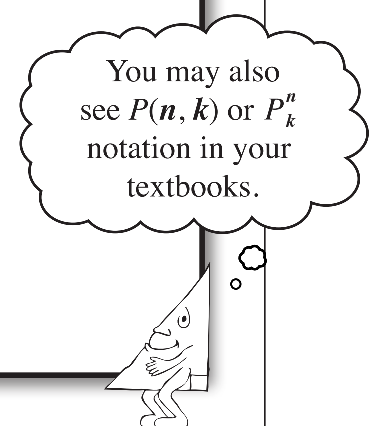
In general, this problem could be represented using the permutation formula.

$$_{\boldsymbol{n}}P_{\boldsymbol{k}} = \frac{\boldsymbol{n}!}{(\boldsymbol{n} - \boldsymbol{k})!}$$

P represents the fact that we are finding a permutation,

$\boldsymbol{n}$ is the total number of objects, and

$\boldsymbol{k}$ is the number of objects we are ordering where $\boldsymbol{k} \leq \boldsymbol{n}$.

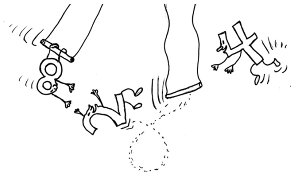


For the problem above, $\boldsymbol{n} = 5$ total racers and $\boldsymbol{k} = 3$ top finishes. So, the number of permutations of the top three finishes is

$$_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 60.$$

Try these:

1) $_7P_2 = \frac{7!}{(7-2)!} =$ ______ **2)** $_9P_4 = \frac{9!}{(9-4)!} =$ ______



• Permutations •

$$_{n}P_{k} = \frac{n!}{(n-k)!}$$

1) $_{10}P_{4} = \frac{10!}{(10-4)!} =$ ______

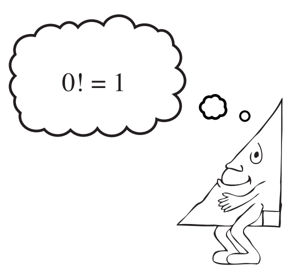
2) $_{8}P_{3} = \frac{8!}{(8-3)!} =$ ______

3) $_{11}P_{2} =$ ______

4) $_{9}P_{3} =$ ______

5) $_{7}P_{1} =$ ______

6) $_{6}P_{4} =$ ______



7) $_{n}P_{n} =$ ______

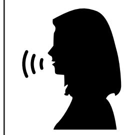
Extending Knowledge

Explain how this is the same as finding the total number of permutations.

____

____

• Combinations •



Direct Teaching

A **COMBINATION** is a selection from a set of objects where order does not matter.

EXAMPLE: Alex (***A***), Bob (***B***), Cat (***C***), and Dylan (***D***) want to play a game that requires teams of 2. How many different teams could there be?

Here, the order of who is chosen for a team does *not* matter as ***AB*** (Alex and Bob being on a team) is the same as ***BA*** (Bob and Alex being on a team). So, for the above scenario, we are looking for the number of **COMBINATIONS** of teams of 2 from a group of 4 friends.

In general, this problem could be represented using the combination formula.

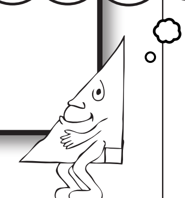
$$_{n}C_{k} = \frac{n!}{k!(n-k)!}$$

C represents the fact that we are finding a combination,

n is the total number of objects, and

k is the number of objects we are choosing where $k \leq n$.

You may also see $C(n, k)$, C^{n}_{k}, or $\binom{n}{k}$ notation in your textbooks. Read aloud, we say "n choose k."



For the problem above, n = 4 friends and k = teams of 2. So, the number of combinations of teams is

$$_{4}C_{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!\,2!} = \frac{4 \times 3 \times \cancel{2!}}{2 \times 1 \times \cancel{2!}} = 6.$$

Try these:

1) $_{5}C_{3} = \frac{5!}{3!(5-3)!} =$ ______

2) $_{7}C_{4} = \frac{7!}{4!(7-4)!} =$ ______

• Combinations •

$$_{n}C_{k} = \frac{n!}{k!(n-k)!}$$

1) $_{9}C_{5} = \frac{9!}{5!(9-5)!} =$ ______

2) $_{8}C_{6} = \frac{8!}{6!(8-6)!} =$ ______

3) $_{12}C_{7} =$ ______

4) $_{5}C_{1} =$ ______

5) $_{15}C_{12} =$ ______

6) $_{13}C_{8} =$ ______

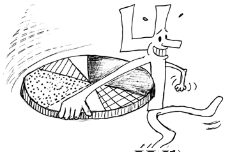
7) $_{n}P_{k} \div {_{k}P_{k}} =$ ________

Explain what you notice about your answer.

____

____

Extending Knowledge



• Permutations and Combinations •

Direct Teaching

Use a Calculator

When solving word problems that involve choosing from a set of objects, we must first figure out whether the problem is asking for a permutation or a combination. We do this by determining whether the order of choosing matters.

Try these:

Example 1: Ethan wants to read a different book from his favorite 10 books on each day of a full week. How many ways can he read his books?

Here, the order of choosing the books to read matters as reading a book on Sunday is different than reading the same book on Monday. Since order matters, we will be setting up a permutation.

$\boldsymbol{n}$ = 10 total books and $\boldsymbol{k}$ = 7 books for each day of the week

$${}_{10}P_7 = \_\_\_\_\_\_\_\_\_\_$$

So, there are ____________ ways that Ethan can read his books.

Example 2: Phoebe can choose 3 items to put on her plate from a buffet spread of mashed potatoes, turkey, chicken, green beans, and carrots. How many ways can she make a plate?

Here, the order of choosing food does not matter as a plate of 3 items does not change based on the order of the items chosen. For example, a plate with turkey, green beans, and carrots (chosen in that order) is the same as a plate with carrots, green beans, and turkey. Since order does not matter, we will be setting up a combination.

$\boldsymbol{n}$ = 5 total food choices and $\boldsymbol{k}$ = 3 items on a plate

$${}_5C_3 = \_\_\_\_\_\_\_\_\_\_$$

So, there are ____________ ways that Phoebe can make a plate.

• Permutations and Combinations •

Explain but do not solve.

1) There are 12 members in a chess club.

a) How many unique matches could be had between the members?

Explain whether you should use a permutation or a combination:

b) If the club wanted to elect a President and a Vice President, how many ways could the positions be filled?

Explain whether you should use a permutation or a combination:

2) There are 8 colors on a paint palette.

a) How many ways could someone paint 3 figures each a different color?

Explain whether you should use a permutation or a combination:

b) How many ways could 3 colors be mixed together?

Explain whether you should use a permutation or a combination:

• Permutations and Combinations •

1) Nathan decides to hang his posters horizontally on his wall: one on the left, one in the middle, and one on the right. If he has 10 posters to choose from to hang, how many arrangements of posters are there?

Does order matter? Explain: ____

____

Set up a permutation or combination expression for this problem:

____

So, there are ____________ arrangements of posters.

2) An ice cream parlor offers a sundae consisting of any 3 flavors out of 12 on the menu. How many different sundaes can be created?

Does order matter? Explain: ____

____

There are ______________ different sundaes.

3) If you cannot repeat letters, how many different 2 letter arrangements can you create from the alphabet?

Does order matter? Explain: ____

____

There are ______________ different 2 letter arrangements.

• Permutations and Combinations •

Use a Calculator

1) Joanne has to narrow down her ranked top 5 favorite movies out of her 10 favorite movies but cannot decide. She decides to choose the movies randomly. How many different ways could her ranked top 5 list come out?

____

2) 12 people are competing on a game show. Teams are divided randomly into 3 even groups. How many different team compositions could there be?

____

3) There are 9 dogs at an animal shelter. If you can only adopt 3 of them, how many different ways could you adopt?

____

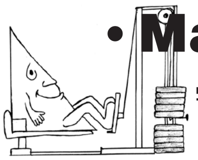
4) Dana wants to travel to 8 different countries but only has time to visit 4. He decides to choose his destinations randomly. How many ways could Dana plan his trip?

____

5) Joseph gets to choose pizza toppings from a selection of 5 meat toppings and 7 veggie toppings. If Joseph chooses 2 random meat toppings and 3 random veggie toppings, how many different ways can he make his pizza?

Extending Knowledge

____



• Mastery Check: Permutations and Combinations •

1) ${}_7P_4 =$ ______

2) ${}_8C_5 =$ ______

3) ${}_{12}P_3 =$ ______

4) ${}_{11}C_8 =$ ______

5) Justin has to take out the trash, do the dishes, mow the lawn, do laundry, and mop the floor. How many ways can Justin finish his chores?

____

6) Sarah can display 4 trophies on her shelf. She has 9 trophies to choose from. How many different arrangements of the trophies are there?

____

7) A deck of cards contains 52 unique cards. How many different combinations of 3 cards are there?

____

Challenge:

8) Philip, Max, Diane, Jason, and Tom are in a bike race. If each biker has an equal chance of winning the race, what is the probability Tom and Jason place in the top two?

____

• Permutations and Probabilities •

Direct Teaching

We can use what we know about permutations to find the number of outcomes of a sample space or an event of that sample space. We can then use that information to find the probability of certain events.

Example: In a race between 10 people that includes Alex, Mary, and Bill, what is the probability that Alex, Mary, and Bill finish in the top 3? Suppose everyone has an equal chance of winning the race.

Since the order of the racers matters, we will use permutations.

Let's first define our sample space S.

S is the set of all the possible top 3 finishes there are in a race out of 10 racers. Now, let's find the number of elements in S.

$$n(\mathrm{S}) = {}_{10}P_3 = 720$$

Next, let's define event A, a subset of S.

Event A is the set of all the possible ways Alex, Mary, and Bill could finish in the top 3. Now, let's find the number of elements in event A.

$$n(\mathrm{A}) = 3! = 6$$

So, the probability that Alex, Mary, and Bill finish in the top 3 is

$$\frac{n(\mathrm{A})}{n(\mathrm{S})} = \frac{6}{720} = \frac{1}{120}.$$

Try this: **Use the scenario above to answer the question.**

1) What is the probability Alex finishes first, Mary finishes second, and Bill finishes third?

____

• Permutations and Probabilities •

Use a Calculator

1) A playlist consisting of 14 different songs is shuffled. If you only enjoy half of the songs, what is the probability that the first 3 songs of the playlist will be songs you enjoy?

Define sample space S: ______________________________________

What is $n(S)$? ____________

Define event A: ____

What is $n(A)$? ____________

So, the probability that the first 3 songs will be songs you enjoy is

$$\frac{n(A)}{n(S)} = \_\_\_\_\_\_\_\_.$$

2) In a class of 30 students including James, Phil, and Tamara, 3 students are chosen randomly to become President, Vice President, and Treasurer. What is the probability that James, Phil, or Tamara could get chosen in any order for these positions?

Define sample space S: ______________________________________

What is $n(S)$? ____________

Define event A: ____

What is $n(A)$? ____________

So, the probability that James, Phil, or Tamara could get chosen is

$$\frac{n(A)}{n(S)} = \_\_\_\_\_\_\_\_.$$

• Permutations and Probabilities •

Use a Calculator

1) A 5-character password is randomly generated from the following characters with no characters repeating: A, B, C, 1, 2, 3, 4, and 5. What is the probability that the first 2 characters of the password will both be numbers?

____

2) A lottery randomly picks a first place winner, second place winner, and third place winner from a pool of 100 people. If you and your 6 family members are in the pool, what is the probability that someone in your family would win each prize?

____

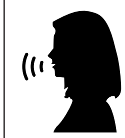
3) A regular keyboard has 26 unique letters on it. If you were to press 4 random letters without repeating any, what is the probability that you would type the word MATH?

____

4) Your school is assigning 7 poems to read, one on each day of a week. If you have already read 3 of them before, what is the probability that you would read a new poem during the first 3 days of the week?

____

• Permutations with Repeated Elements •



Sometimes when we are finding permutations, there may be repeated elements in the set.

Example: How many permutations are there of the letters of the word MISSISSIPPI?

If we were to rearrange the letters of the word MISSISSIPPI, there would be duplicate permutations since there are 4 I's, 4 S's, and 2 P's. However, we can divide out these duplicates from the total number of permutations to find the correct number of unique permutations.

> The number of permutations with repetition is
>
> $$\frac{n!}{x_1!\, x_2! \ldots x_r!},$$
>
> where n represents the total numbers of elements, x_1 represents the number of repetitions of the first unique element, and x_r such that $r \le n$ represents the number of repetitions of the last unique element in the set.

In our above example, n = 11 letters in MISSISSIPPI, x_1 = the number of repetitions for M (1), x_2 = the number of repetitions for I (4), x_3 = the number of repetitions for S (4), and x_4 = the number of repetitions for P (2).

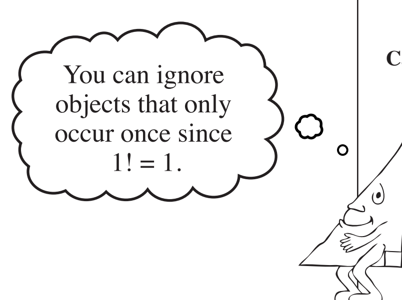
So, the number of permutations of the letters of the word MISSISSIPPI is

$$\frac{11!}{1!\, 4!\, 4!\, 2!} = 34{,}650.$$

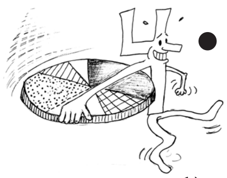
Try this:

1) How many permutations are there of the letters of the word BARBARIAN?

n = ______ x_1 = ______ x_2 = ______ x_3 = ______



The number of permutations is ______________.



• Permutations with Repeated Elements •

Use a Calculator

1) A class has 6 girls and 7 boys. If they all lined up, how many ways could girls and boys be arranged?

n = ______ x_1 = ______ x_2 = ______

The number of permutations is ________________.

2) How many permutations are there of the letters of the word TEEPEE?

n = ______ x_1 = ______

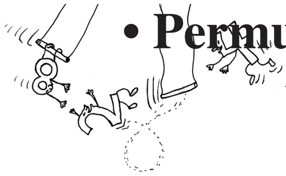
The number of permutations is ________________.

3) A grocer has 5 apples, 4 bananas, and 3 pears. If he were to arrange the fruits in a line, how many ways could he arrange the fruits?

____

4) How many permutations are there of the letters of the word MATHNASIUM?

____



• Permutations with Repeated Elements and Probabilities •

Direct Teaching

EXAMPLE: What is the probability that a random arrangement of the word MISSISSIPPI will have all the S's, I's, and P's next to each other?

S is the set of all the permutations of the word MISSISSIPPI. We know that the total number of permutations of the word MISSISSIPPI is $\frac{11!}{1!\,4!\,4!\,2!}$ = 34,650. So, $n(\text{S})$ = 34,650.

Event A, a subset of S, is the set of all permutations of the word MISSISSIPPI where all the S's, I's and P's are next to each other.

SSSS IIII PP M and M SSSS IIII PP are two examples of permutations where all the S's, I's, and P's are next to each other.

So, we can think of SSSS, IIII, PP, and M as 4 separate elements in A. Finding the number of permutations of the 4 elements will give us $n(\text{A})$.

$$n(\text{A}) = 4! = 24$$

Therefore, the probability that a random arrangement of MISSISSIPPI will have all the S's, I's, and P's next to each other is

$$\frac{n(\text{A})}{n(\text{S})} = \frac{24}{34{,}650} = \frac{4}{5{,}775}.$$

Try this:

1) What is the probability that a random arrangement of the word MISSISSIPPI will start with the letter M?
HINT: How many ways can we arrange ISSISSIPPI?

____

• Permutations with Repeated Elements and Probabilities •

Use a Calculator

1) Cathy owns 3 cats and 3 dogs. If she feeds them in a random order, what is the probability that she would alternate between cats and dogs?

S is the set of all permutations of 3 cats and 3 dogs. So, $n(\text{S})$ = __________.

List all the ways cats (c) and dogs (d) could alternate.

____

So, $n(\text{A})$ = __________.

Therefore, the probability that she would alternate is _______.

2) If you randomly rearrange the letters of the word DESSERTS, what is the probability that you would get the word STRESSED?
Hint: How many times does STRESSED occur in the sample space?

____

3) You have 3 apples, 2 oranges, 1 banana, and 1 pear to eat. If you eat all the fruit in a random order, what is the probability you would eat all the apples in a row?
Hint: Group the 3 apples as 1 element.

____

Extending Knowledge

4) The letters of the word MISSISSIPPI are on tiles in a bag. If you randomly pick 4 letters from the bag, what is the probability that you get MISS in that order?
Hint: How many ways can you choose the first I? First S? Second S?

____

Direct Teaching

We can also use what we know about combinations to find the number of elements in a set or subset of possible outcomes. We can then use that information to find the probability of certain events.

Example: There are 5 red marbles and 7 blue marbles in a bag. If 4 marbles are chosen randomly, what is the probability that all marbles will be blue?

Since the order of choosing marbles does not matter, we will use combinations.

S is the set of all possible ways we can choose 4 marbles out of 12.

Now, let's find the number of elements in set S.

$$n(\text{S}) = {}_{12}C_4 = 495$$

Event A is the set of all possible ways we can choose 4 blue marbles.

Since the only way we can choose 4 blue marbles is to choose all of them out of the 7 total blue marbles, the number of ways we can choose 4 blue marbles is ${}_7C_4$.

$$n(\text{A}) = {}_7C_4 = 35$$

So, the probability that all marbles will be blue is

$$\frac{n(\text{A})}{n(\text{S})} = \frac{35}{495} = \frac{7}{99}.$$

Try this: **Use the scenario above to answer the question.**

1) If we choose 4 marbles, what is the probability that all marbles will be red?

____

• Combinations and Probabilities •

Use a Calculator

1) In a standard shuffled 52 card deck, if you pull 2 random cards, what is probability that both cards will be aces?

Define sample space S: ____

What is n(S)? ____________

Define event A: ____

What is n(A)? ____________

So, the probability that both cards will be aces is

$$\frac{n(A)}{n(S)} = \_\_\_\_\_\_\_\_.$$

2) In a class of 12 girls and 15 boys, if 4 students are chosen at random, what is the probability that all the students will be boys?

Define sample space S: ____

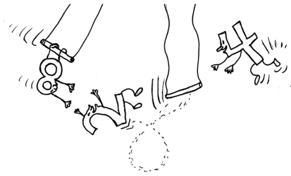
What is n(S)? ____________

Define event A: ____

What is n(A)? ____________

So, the probability that all the students will be boys is

$$\frac{n(A)}{n(S)} = \_\_\_\_\_\_\_\_.$$



• Combinations and Probabilities •

Use a Calculator

1) An animal shelter has 6 cats and 8 dogs. If you adopt 3 pets randomly, what is the probability that all of them will be dogs?

____

2) If you choose 4 digits randomly without repetition from 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, what is the probability that you choose the numbers 0, 1, 2, and 3? **HINT**: How many times does any combination of 4 numbers occur in the sample space?

____

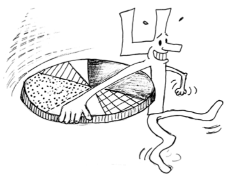
3) A basketball squad of 5 is randomly created from a group of 10 boys and 8 girls. What is the probability that the team will consist of all girls?

____

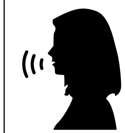
Extending Knowledge

4) You have 4 fiction novels, 5 nonfiction novels, and 3 comic books you want to read. If you pick 4 books at random, what is the probability that all the books will be comic books?

____



• Combinations and Probabilities •



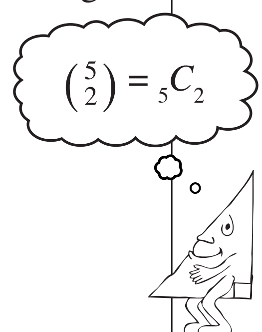
We must pay attention when solving probability word problems that involve words like “exactly,” “at least,” “at most,” “no more than,” or “no less than.”

Example: You flip a coin 5 times. What is the probability you will get exactly 2 heads?

S is the set of all the possibilities of coin outcomes in 5 flips. For example, HHHHH and HHHHT are two possible outcomes. Since there are 2 possibilities per flip, we find that $n(S) = 2^5 = 32$.

Event A is the set of all possibilities of getting exactly 2 heads out of 5 flips.

Let’s relate each coin flip to being chosen (heads) or not chosen (tails). So, the number of ways the coin can show 2 heads out of 5 flips is the same is finding $\binom{5}{2}$. Or, $n(A) = \binom{5}{2} = 10$.



Therefore, the probability that you will get exactly 2 heads is

$$\frac{n(A)}{n(S)} = \frac{10}{32} = \frac{5}{16}.$$

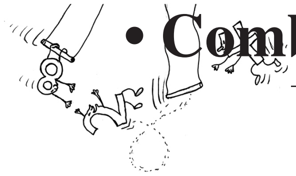
Try these:

1) You flip a coin 6 times. What is the probability you will get exactly 4 heads?

____

2) You flip a coin 3 times. What is the probability you will get exactly 3 heads?

____



• Combinations and Probabilities with Addition •

Use a Calculator

For problems that involve words like "at most" or "at least," we must add combinations to find the number of elements in our events.

Example: You flip a coin 5 times. What is the probability you will get at least 3 heads?

Again, S is the set of all the possibilities of coin outcomes in 5 flips.

$$\text{So, } n(\text{S}) = 2^5 = 32.$$

Event B is the set of all possibilities of coin outcomes with at least 3 heads. So, event B consists of three separate non-overlapping subsets of S: the event of getting exactly 3 heads, exactly 4 heads, and exactly 5 heads.

We can add all of these combinations together to find the number of elements of B.

$$n(\text{B}) = \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 10 + 5 + 1 = 16$$

Therefore, the probability that you will get at least 3 heads is

$$\frac{n(\text{B})}{n(\text{S})} = \frac{16}{32} = \frac{1}{2}.$$

Try this:

1) You flip a coin 6 times. What is the probability you will get at most 3 heads?
Hint: Find the number of combinations of getting 0 heads, exactly 1 head, exactly 2 heads, and exactly 3 heads.

____

• Probability Problem Solving •

Use a Calculator

You flip a coin 8 times.

1) What is the probability you will get exactly 3 heads?

____

2) What is the probability you will get at most 3 heads?

____

3) What is the probability you will get at least 3 heads?
Hint: Use your answers from #1 and #2.

____

Extending Knowledge

4) At a party, there are 8 girls and 6 boys. If half the attendees are chosen at random to join a raffle, what is the probability that exactly 4 of them will be boys?
Hint: If we are choosing 4 boys, we are also choosing 3 girls.

____

5) A piggy bank has 4 quarters, 3 dimes, and 2 nickels. If you choose 2 coins from the piggy bank at random, what is the probability you will have at least 30 cents?

____

• Probability Problem Solving •

Use a Calculator

1) An animal shelter has 5 cats and 6 dogs. If you adopt 4 pets randomly, what is the probability that 2 of them will be cats and 2 will be dogs?

____

2) A bag contains 2 red marbles, 2 green marbles, and 2 blue marbles. If you arranged all the marbles in a row randomly, what is the probability that all the red and blue marbles will be together?
HINT: List all the ways red and blue can be together.

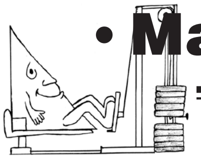
____

3) You roll a fair six-sided die 3 times. What is the probability that you will get at most a 2 on all 3 rolls?

____

4) If you draw 5 cards randomly from a standard deck of 52 cards, what is the probability that all 5 cards will have the same suit?

____



• Mastery Check: Probability Problem Solving •

Use a Calculator

1) In a race between 12 people that includes you and your 5 friends, what is the probability that all of you will finish in the top half of the race? Suppose everyone has an equal chance to win the race.

____

2) A playlist consists of 5 rap songs, 5 rock songs, and 5 electronic songs. What is the probability that 5 songs chosen at random from the playlist will all be rap songs?

____

3) What is the probability that a random arrangement of the word SUMMER will start and end with an M?

____

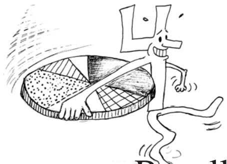
4) You flip a coin 6 times. What is the probability you will get at least 3 heads?

____

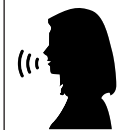
Challenge:

5) In a race between 9 people that includes Bill, Turk, and Fran, what is the probability that Bill and Turk or Bill and Fran finish in the top 3? Suppose everyone has an equal chance to win the race.

____



• Inverse Variation •



Direct Teaching

Recall, an *inverse variation* relationship is when one quantity increases, the other decreases proportionally. Let's take a look a few examples of when one quantity is inversely proportional to two or more other values.

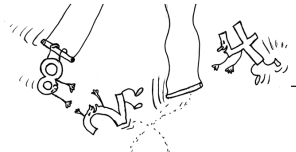
The table below shows the relationship between inverse variation statements with more than two variables and their equations.

Statement	Equation	Constant of proportionality
The value of z varies inversely as the values of x and y.	$z = \frac{k}{xy}$	k
The value of p varies inversely as the square of q and the square root of r.	$p = \frac{k}{q^2\sqrt{r}}$	k
The Inverse Square Law states the force between an object and earth is inversely proportional to the square of the distance between the object and earth's center.	$F = \frac{k}{d^2}$	k

NOTE: The value of the constant of proportionality is not stated in word form.

***Try this*: Fill in the missing parts in the table below.**

Statement	Equation	Constant of proportionality
The value of r varies inversely as the values of s and t.		k
	$a = \frac{3}{bc^2}$	
	$y = \frac{k}{xz^3}$	



• Inverse Variation •

Solve the following inverse variation problems.

1) The value of $\boldsymbol{a}$ varies inversely as the values of $\boldsymbol{b}$ and $\boldsymbol{c}$. If $\boldsymbol{a} = 3$ when $\boldsymbol{b} = 5$ and $\boldsymbol{c} = 2$, what is the value of $\boldsymbol{a}$ when $\boldsymbol{b} = 3$ and $\boldsymbol{c} = 2\frac{1}{2}$?

What is the constant of proportionality? __________

Write the inverse variation equation. ____________________

$\boldsymbol{a}$ = __________

2) The value of $\boldsymbol{r}$ varies inversely as the cube of $\boldsymbol{s}$ and the square root of $\boldsymbol{t}$. If $\boldsymbol{r} = \frac{1}{2}$ when $\boldsymbol{s} = 2$ and $\boldsymbol{t} = 36$, what is the value of $\boldsymbol{r}$ when $\boldsymbol{s} = 3$ and $\boldsymbol{t} = 16$?

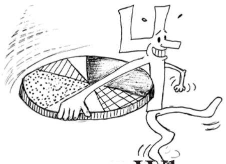
Write the inverse variation equation. ____________________

$\boldsymbol{r}$ = __________

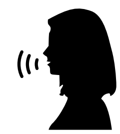
3) The value of $\boldsymbol{l}$ varies inversely as the cube root of $\boldsymbol{m}$ and the square of $\boldsymbol{n}$. If $\boldsymbol{l} = 2$ when $\boldsymbol{m} = 64$ and $\boldsymbol{n} = 3$, what is the value of $\boldsymbol{m}$ when $\boldsymbol{l} = 1\frac{1}{2}$ and $\boldsymbol{n} = 4$?

Write the inverse variation equation. ____________________

$\boldsymbol{m}$ = __________



• Combined Variation •



Direct Teaching

When two quantities increase or decrease proportionally, we have a *direct variation*. When one quantity increases and the other decreases proportionally, we have an *inverse variation*. There are also some situations where a quantity will vary directly with some value(s) and inversely with others. This is called **COMBINED VARIATION**.

The situation below shows how a direct variation equation and an inverse variation equation about the same scenario can be combined into one equation.

EXAMPLE: A company's sales, s, are directly proportional to their marketing budget, m, and inversely proportional to their product cost, c.

The first part of this statement tells us that s varies directly with m and gives us the following direct variation equation.

$$s = km$$

In contrast, the company's sales, s, decrease as the cost, c, of their product increases. This is an inverse relationship and we say that s varies inversely with c.

$$s = \frac{k}{c}$$

We can combine these two equations to form one combined variation equation.

$$s = \frac{km}{c}$$

Try this: **Fill in the missing parts in the table below.**

Statement	Equation	Constant of proportionality
The value of y varies directly as the value of z and inversely with the value of x.	$y = \frac{kz}{x}$	k
	$a = \frac{3bc}{d^2}$	
The volume of gas varies directly as the temperature and inversely as the pressure.		

• Combined Variation •

Direct Teaching

Let's take a look at how we can use combined variation equations to solve for unknown values.

Example: The value of z varies directly as the value of x and inversely as the value of y. If $z = 27$ when $x = 3$ and $y = 4$, what is the value of z when $x = 17$ and $y = 12$?

Steps to Solve:

Step 1: Write the combined variation equation. Use any appropriate letter to represent the constant of proportionality.	z varies directly as the value of x and inversely with the value of y, written as an equation, is $z = \frac{kx}{y}$ where k is the constant of proportionality.
Step 2: Substitute the given x, y, and z values to find the constant of proportionality.	If $z = 27$, $x = 3$, and $y = 4$, then $27 = \frac{k(3)}{(4)}$ and $k = 36$.
Step 3: Rewrite the combined variation equation with the constant of proportionality.	$z = \frac{kx}{y} \Rightarrow z = \frac{36x}{y}$.
Step 4: Substitute the given x and y values to find their corresponding z value.	If $x = 17$ and $y = 12$, then $z = \frac{36(17)}{(12)}$ and $z = 51$. So, $z = 51$ when $x = 17$ and $y = 12$.

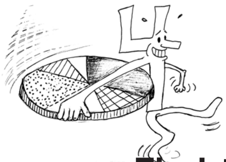
Try this:

1) The value of c varies directly as the value of a and inversely with the value of b, and $c = 39$ when $a = 13$ and $b = 5$. What is the value of a when $b = 18$ and $c = 5$?

What is the constant of proportionality? ___________

Write the combined variation equation. ______________________

a = ___________



• Combined Variation •

☆

Find the constant of proportionality, variation equation, and the unknown value.

1) The value of w varies jointly as the values of x and y and inversely as the square of z. If $w = 30$ when $x = 5$, $y = 2$, and $z = 3$, what is the value of w when $x = 4$, $y = 9$, and $z = 9$?

What is the constant of proportionality? __________

Write the combined variation equation. ____________________

$w =$ __________

2) The value of p varies directly as the square of q and inversely as the square root of r. If $p = 60$ when $q = 6$ and $r = 81$, what is the value of p when $q = 8$ and $r = 144$?

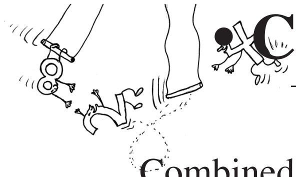
Write the combined variation equation. ____________________

$p =$ __________

3) The value of a varies jointly as the values of b and c and inversely as the square root of d. If $a = 15$ when $b = 2$, $c = 27$, and $d = 36$, what is the value of a when $b = 21$, $c = 13$, and $d = 49$?

Write the combined variation equation. ____________________

$a =$ __________



• Combined Variation Problem Solving •

Direct Teaching

Combined variation equations are also used to solve various real world problems.

Example: A person's BMI (body mass index) varies directly as the person's weight in pounds and inversely as the square of the person's height in inches. A person who weighs 140 lbs and is 70 inches tall has a BMI of 20. What is the BMI of a person who is 64 inches tall and weighs 118 lbs?

Steps to Solve:

Step 1: Write the variation equation. Use appropriate letters to represent the variables.	$b = \frac{kw}{h^2}$ where k is the constant of proportionality.
Step 2: Substitute the given b, w, and h values to find the constant of proportionality.	If $b = 20$, $w = 140$, and $h = 70$, then $20 = \frac{k(140)}{(70)^2}$ and $k = 700$.
Step 3: Rewrite the variation equation with the constant of proportionality.	$b = \frac{kw}{h^2} \Rightarrow b = \frac{700w}{h^2}$.
Step 4: Substitute the given values to find the unknown.	If $w = 118$ and $h = 64$, then $b = \frac{700(118)}{(64)^2}$ and $b = 20.2$. A person who weighs 118 lbs and is 64 inches tall has a BMI of 20.2.

Try this:

1) The frequency of vibrations of a string varies directly as the square root of the tension and inversely with the length of the string. A 3 ft long string vibrates 40 times per second when the tension is 25 pounds. What is the frequency in vibrations per second of a 4 ft long string with a tension of 36 pounds?

What is the combined variation equation? ________________

____

• Combined Variation Problem Solving •

Find the constant of proportionality, variation equation, and the unknown value. Leave the constant of variation in fraction form.

Use a Calculator

1) The volume of a gas varies directly as the temperature and inversely as the pressure. When the temperature is 276 ºK and the pressure is 23 lbs/cm^2, the volume of the gas is 184 cm^3. What is the volume of the same gas when the temperature is 270 ºK and the pressure is 30 lbs/cm^2?

What is the combined variation equation? _________________

Volume = _________________

2) The maximum load a cylindrical column with a circular cross-section can hold varies directly as the fourth power of the diameter and inversely as the square of the height. A 10-meter column, 2 meters in diameter, will support 60 metric tons. How many metric tons can be supported by a column 25 meters high and 3 meters in diameter?

What is the combined variation equation? _________________

Weight = _________________

3) The time required to process an order varies directly as the number of items in the order and inversely with the number of workers. If 3,000 items can be processed by 5 workers in 8 hours, then how many workers are needed to process 9,000 items in 6 hours?

What is the combined variation equation? _________________

Number of workers = _________________

• Combined Variation •

Use a Calculator

Solve. Round answers to the nearest hundredth as necessary.

1) The value of r varies directly as t squared and inversely as s. If $r = 192$ when $t = 8$ and $s = 3$, what is the value of s when $t = 9$ and $r = 486$?

____

2) The speed of a bicycle varies jointly as the number of revolutions per minute that the pedals turn and the number of teeth on the front gear. The speed also varies inversely as the number of teeth on the back gear. Someone who pedals 80 rpm on a bicycle that has 35 teeth in its front gear and 15 in the back gear travels at 14.5 mph. If they increased their pedaling to 100 rpm on the same bike, what would be their new speed?

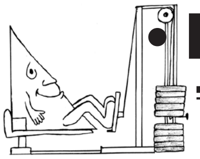
____

3) The value of f varies jointly as the value of g and the square root of h, and inversely as the square of j. If $f = 2$ when $g = {}^{-}7$, $h = 81$, and $j = 6$, what is the value of f when $g = 28$, $h = 64$, and $j = 7$?

____

4) The value of a varies directly as the value of b and inversely as the cube root of c. If $a = 4$ when $b = {}^{-}32$ and $c = 125$, what is the value of a when $b = 56$ and $c = 27$?

____



• Mastery Check: Combined Variation •

Solve.

1) The value of w varies jointly as the values of x and y, and inversely as the square of z. If $w = 30$ when $x = 5$, $y = 2$, and $z = 3$, what is the value of w when $x = 4$, $y = 9$, and $z = 9$?

____

2) The stopping distance of a car varies directly as the speed of the car and inversely as the friction of the road. A car traveling at 20 mph takes 50 feet to stop on a road with a friction value of 2. How many feet will it take the same car traveling 35 mph to stop on a road with a friction value of 2.5?

____

Challenge:

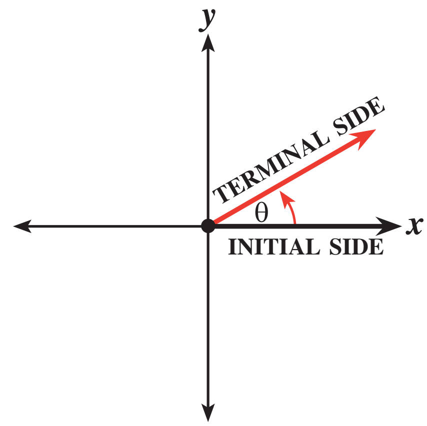
3) Suppose y varies jointly as z and the cube of x and inversely with the square of r. What is the effect on y when x is doubled and r is halved?

____

• Angles on the Coordinate Plane •

Direct Teaching

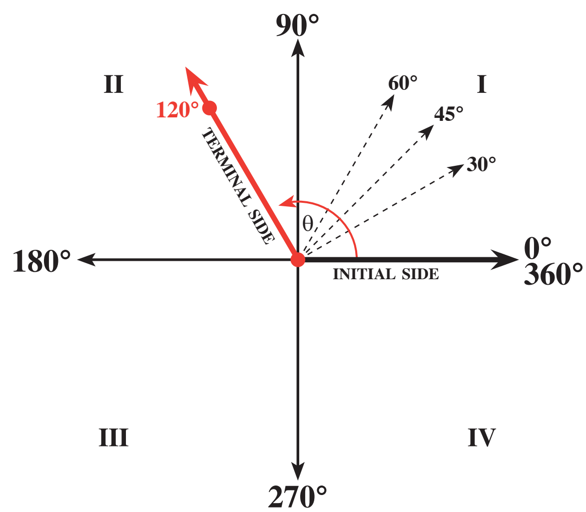
An **ANGLE** on the coordinate plane whose **INITIAL SIDE** lies on the positive x-axis and whose **VERTEX** lies on the origin is in **STANDARD POSITION**. We can measure an angle in **STANDARD POSITION** based on the amount of rotation from its **INITIAL SIDE** to its **TERMINAL SIDE**.



EXAMPLE: Mark and label a 120° angle on the coordinate plane.

Whenever we mark an angle on the coordinate plane, we can assume the angle is in **STANDARD POSITION**. Therefore, we only need to mark the angle's **TERMINAL SIDE**.

We can use certain benchmark angles in Quadrant I to help us mark angles in the other quadrants. These benchmark angles are marked on the coordinate plane on the right: 30°, 45°, and 60°.



In this case, 120° is in Quadrant II between 90° than 180°. Furthermore, 120° is 60° less than 180°, so we can mark the angle 60° away from the negative x-axis.

Try these: **Mark and label the angles on the coordinate plane above.**

1) 135°

2) 150°

• Angles on the Coordinate Plane •

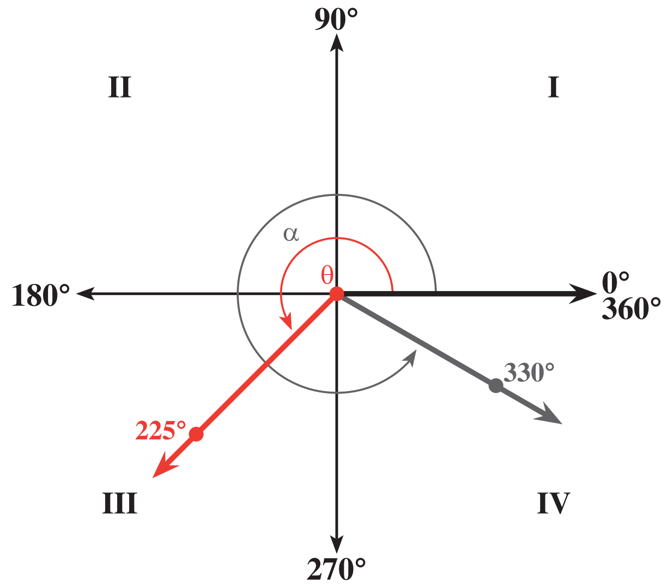
Direct Teaching

Let's investigate how to use our benchmark angles 30°, 45°, and 60° to mark angles in the other quadrants.

Example: Mark and label a 225° angle and a 330° angle on the coordinate plane.

225° is in Quadrant III between 180° and 270°. 225° is 45° more than 180°, so we can mark the angle 45° away from the negative ***x***-axis and label it θ.

Meanwhile, 330° is in Quadrant IV between 270° and 360°. 330° is 30° less than 360°, so we can mark the angle 30° away from the positive ***x***-axis and label it α to differentiate it from the 225° angle.



Try these:

1) Which quadrant is 315° in? ____________________

What benchmark angle(s) can you use to mark 315°? ____________________

Mark and label 315° on the coordinate plane above.

2) Which quadrant is 240° in? ____________________

What benchmark angle(s) can you use to mark 240°? ____________________

Mark and label 240° on the coordinate plane above.

• Angles on the Coordinate Plane •

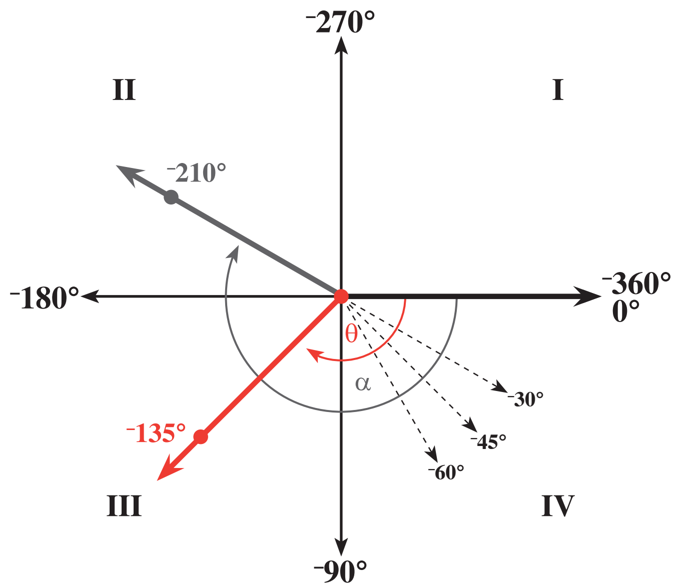
Direct Teaching

We can also use benchmark angles to mark negative angles on the coordinate plane. However, instead of using 30°, 45°, and 60° in Quadrant I, we use ⁻30°, ⁻45°, and ⁻60° in Quadrant IV.

Example: Mark and label a ⁻135° angle and a ⁻210° angle on the coordinate plane.

⁻135° is in Quadrant III between ⁻90° and ⁻180°. ⁻135° is 45° more than ⁻180°, so we can mark the angle 45° away from the negative ***x***-axis and label it θ.

Meanwhile, ⁻210° is in Quadrant II between ⁻180° and ⁻270°. ⁻210° is 30° less than ⁻180°, so we can mark the angle 30° away from the negative ***x***-axis and label it α.



Try these:

1) Which quadrant is ⁻225° in? ____________________

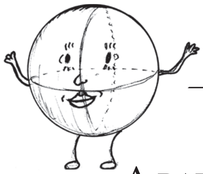
Which benchmark angle(s) can you use to mark ⁻225°? ________________

Mark and label ⁻225° on the coordinate plane above.

2) Which quadrant is ⁻330° in? ____________________

Which benchmark angle(s) can you use to mark ⁻330°? ________________

Mark and label ⁻330° on the coordinate plane above.



• Marking Radians Using Degrees •

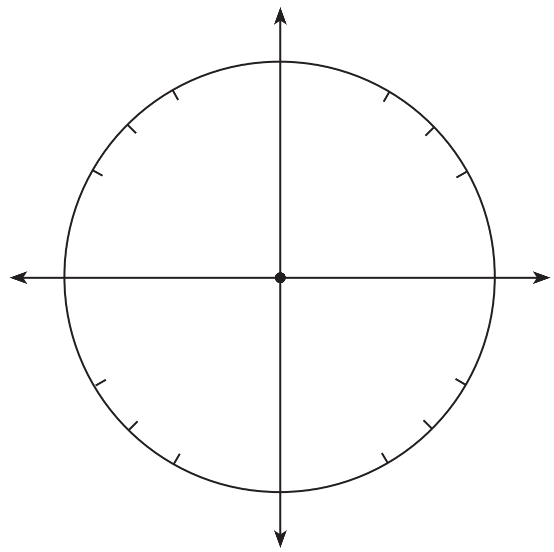
Direct Teaching

A **RADIAN** is another unit of measurement of rotation. We use the **UNIT CIRCLE** with a radius of 1 to mark **RADIANS**, and there are 2π **RADIANS** in the **UNIT CIRCLE**.

We can use our understanding of angles to mark **RADIANS** on the **UNIT CIRCLE**.

Try this:

EXAMPLE: Mark and label $\frac{7\pi}{6}$ on the **UNIT CIRCLE** by converting to degrees.

STEP 1: Convert the **RADIANS** to degrees.	$\frac{7\pi \text{ radians}}{6} \times \frac{180 \text{ degrees}}{\pi \text{ radians}} =$ ________
STEP 2: Mark and label $\frac{7\pi}{6}$ on the **UNIT CIRCLE** using this new degree measurement. Which benchmark angle(s) can you use? ____________________	

1) What is $\frac{\pi}{4}$ in degrees? $\frac{\pi}{4} \times \frac{180}{\pi} =$ ____________

Use the degree measurement to mark $\frac{\pi}{4}$ on the **UNIT CIRCLE** above.

2) What is $\frac{2\pi}{3}$ in degrees? $\frac{2\pi}{3} \times \frac{180}{\pi} =$ ____________

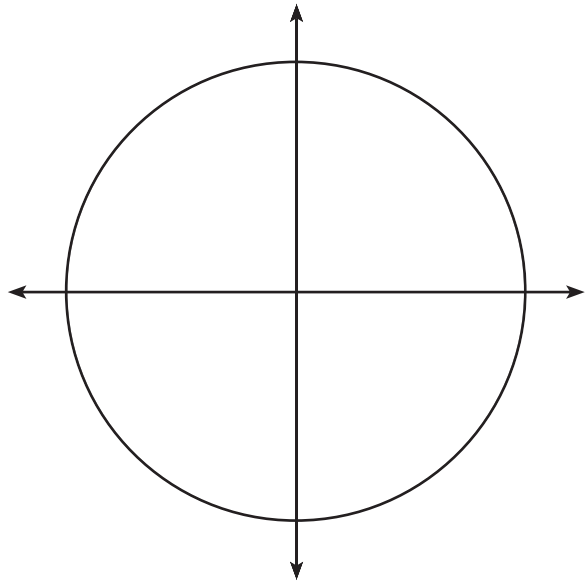
Use the degree measurement to mark $\frac{2\pi}{3}$ on the **UNIT CIRCLE** above.

• Marking Degrees and Radians •

1) Which quadrant is 210° in?

____

Mark and label 210° on the circle.



2) Which quadrant is $^{-}300°$ in?

____

Mark and label $^{-}300°$ on the circle.

Convert the following radian measurements to degrees. Then, mark and label positions for the angles above.

3) $\frac{\pi}{6}$ radians = ___________°

4) $-\frac{\pi}{4}$ radians = ___________°

5) $-\frac{2\pi}{3}$ radians = ___________°

6) $\frac{5\pi}{4}$ radians = ___________°

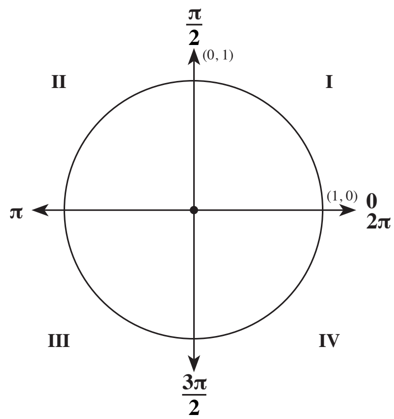
7) $\frac{3\pi}{2}$ radians = ___________°

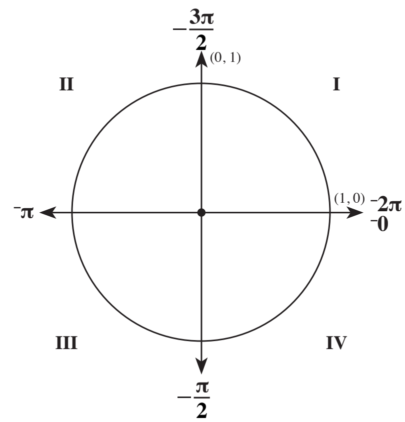
8) $-\frac{5\pi}{4}$ radians = ___________°

• Radians Around the Unit Circle •

Direct Teaching

We compare **RADIAN** measurements to the **QUADRANTAL ANGLES** (the angles that define the boundaries of the quadrants) to determine which quadrants they lie in.





EXAMPLE 1: Which quadrant is $\frac{4\pi}{3}$ in?

$\frac{4\pi}{3}$ is $\frac{4}{3}$ of π. Let's use the **UNIT CIRCLE** with positive **RADIANS**: $\frac{4}{3}$ is between $\frac{3}{2}$ and 1, so $\frac{4\pi}{3}$ is between π and $\frac{3\pi}{2}$ in Quadrant III.

EXAMPLE 2: Which quadrant is $-\frac{7\pi}{4}$ in?

$-\frac{7\pi}{4}$ is $-\frac{7}{4}$ of π. Let's use the **UNIT CIRCLE** with negative **RADIANS**: $-\frac{7}{4}$ is between $-\frac{3}{2}$ and ⁻2, so $-\frac{7\pi}{4}$ is between $-\frac{3\pi}{2}$ and ⁻2π in Quadrant I.

Try these: **Use the UNIT CIRCLES above to determine which quadrant each radian measurement is in.**

1) $\frac{11}{6}$ is between $\frac{3}{2}$ and ______, so $\frac{11\pi}{6}$ is between $\frac{3\pi}{2}$ and ______.

Therefore, $\frac{11\pi}{6}$ is in Quadrant ______.

2) $-\frac{2}{3}$ is between ______ and ______, so $-\frac{2\pi}{3}$ is between ______ and ______.

Therefore, $-\frac{2\pi}{3}$ is in Quadrant ______.

3) $\frac{5\pi}{6}$ is in Quadrant ______.

4) $-\frac{5\pi}{4}$ is in Quadrant ______.

• Radians Around the Unit Circle •

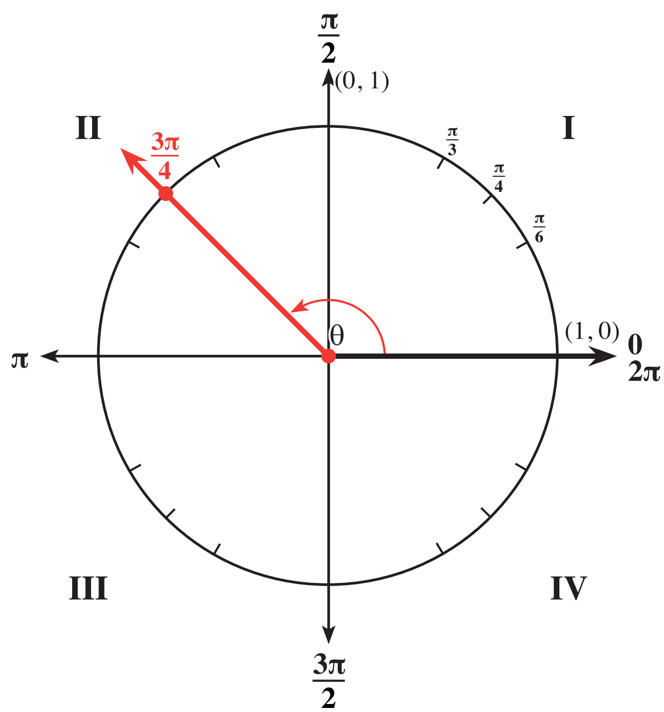
Direct Teaching

To mark **RADIAN** measurements around the **UNIT CIRCLE**, we use the same benchmark angles in Quadrant I that we used for degree measurements. Our benchmark angles are $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°).

EXAMPLE: Mark and label $\frac{3\pi}{4}$ on the **UNIT CIRCLE**.

$\frac{3\pi}{4}$ is in Quadrant II between $\frac{\pi}{2}$ and π.

We can use our benchmark angle $\frac{\pi}{4}$ to mark $\frac{3\pi}{4}$. $\frac{3\pi}{4}$ is $\frac{\pi}{4}$ less than π, so we mark the angle $\frac{\pi}{4}$ away from the negative x-axis.



Try these: **Mark and label the angles on the coordinate plane above.**

1) $\frac{2\pi}{3}$

2) $\frac{5\pi}{6}$

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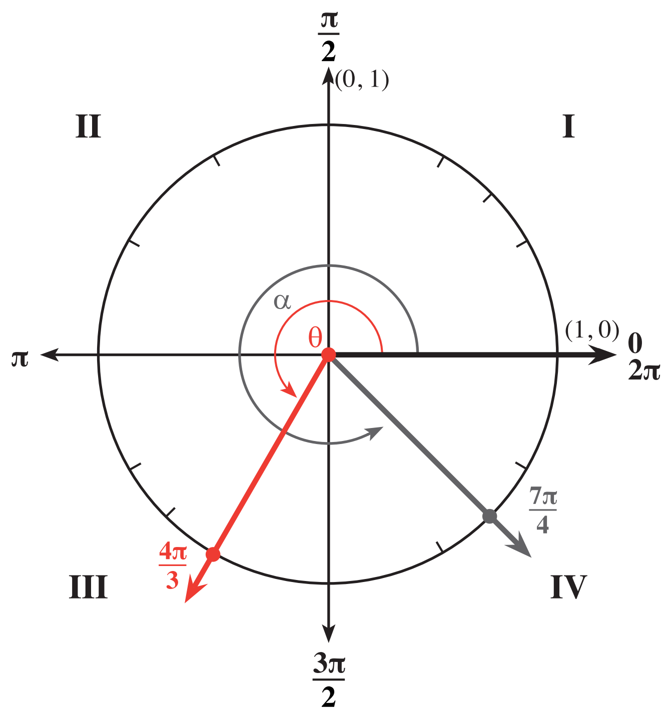
Direct Teaching

EXAMPLE: Mark and label $\frac{4\pi}{3}$ and $\frac{7\pi}{4}$ on the UNIT CIRCLE.

$\frac{4\pi}{3}$ is in Quadrant III between π and $\frac{3\pi}{2}$. Furthermore, $\frac{4\pi}{3}$ is $\frac{\pi}{3}$ more than π, so we can mark the angle $\frac{\pi}{3}$ away from the negative x-axis and label it θ.

Meanwhile, $\frac{7\pi}{4}$ is in Quadrant IV between $\frac{3\pi}{2}$ and 2π. $\frac{7\pi}{4}$ is $\frac{\pi}{4}$ less than 2π, so we can mark the angle $\frac{\pi}{4}$ away from the positive x-axis and label it α.



Try these:

1) Which quadrant is $\frac{5\pi}{3}$ in? ____________________

Which benchmark angle can you use to mark $\frac{5\pi}{3}$? ____________________

Mark and label $\frac{5\pi}{3}$ on the circle above.

2) Which quadrant is $\frac{5\pi}{4}$ in? ____________________

Which benchmark angle can you use to mark $\frac{5\pi}{4}$? ____________________

Mark and label $\frac{5\pi}{4}$ on the circle above.

• Negative Radians Around the Unit Circle •

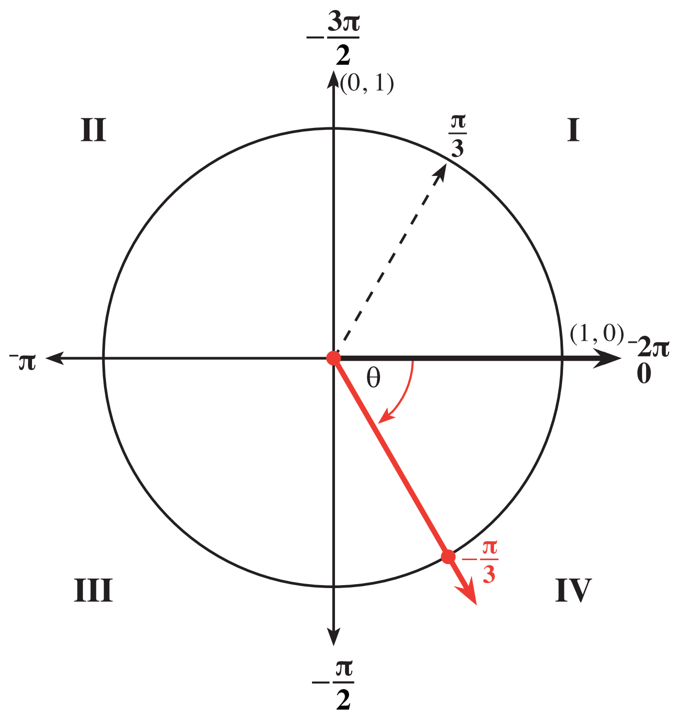
Let's consider how changing the direction of rotation around the UNIT CIRCLE changes our benchmark angles.

Direct Teaching

EXAMPLE: Mark and label $-\frac{\pi}{3}$ on the UNIT CIRCLE.

While $\frac{\pi}{3}$ is in Quadrant I, $-\frac{\pi}{3}$ is in Quadrant IV because of its clockwise rotation. Therefore, it is between 0 and $-\frac{\pi}{2}$, and we can mark the angle $\frac{\pi}{3}$ away from the positive x-axis.

Just like we can use $\frac{\pi}{3}$ as a benchmark angle, we can also use $-\frac{\pi}{3}$ as a benchmark angle for negative angles. In fact, $-\frac{\pi}{3}$ is a reflection of $\frac{\pi}{3}$ over the x-axis.



Try these:

1) Mark and label $-\frac{\pi}{4}$ on the UNIT CIRCLE above.

2) Mark and label $-\frac{\pi}{6}$ on the UNIT CIRCLE above.

• Negative Radians Around the Unit Circle •

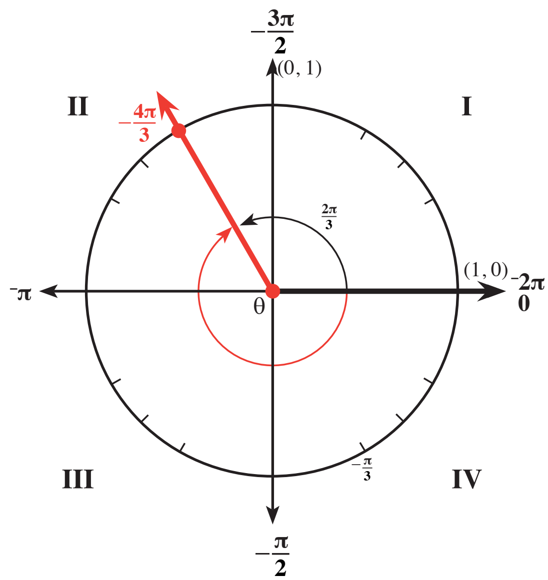
Direct Teaching

When marking negative angles, we can either use negative benchmark angles, or we can use **COTERMINAL ANGLES**. **COTERMINAL ANGLES** can be useful when a positive angle is easier to visualize than the negative angle that shares its **INITIAL** and **TERMINAL SIDES**.

EXAMPLE: Mark and label $-\frac{4\pi}{3}$ on the **UNIT CIRCLE**.

$-\frac{4\pi}{3}$ is in Quadrant II between $^-\pi$ and $-\frac{3\pi}{2}$.

We can use our benchmark angle of $-\frac{\pi}{3}$ to accurately mark $-\frac{4\pi}{3}$. $-\frac{4\pi}{3}$ is $\frac{\pi}{3}$ less than $^-\pi$, so we can mark the angle $\frac{\pi}{3}$ away from the negative x-axis.



Alternatively, we can find a **COTERMINAL ANGLE** of $-\frac{4\pi}{3}$:

$$-\frac{4\pi}{3} + 2\pi = \frac{2\pi}{3}$$

We can therefore mark $\frac{2\pi}{3}$ on the **UNIT CIRCLE** and label it as $-\frac{4\pi}{3}$ instead.

Try this:

1) Which quadrant is $-\frac{5\pi}{6}$ in? ____________________

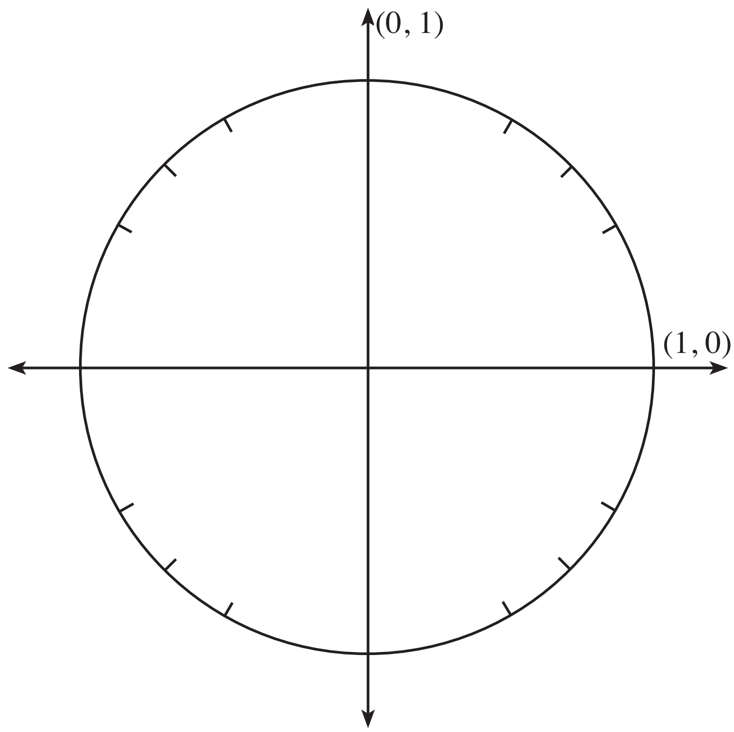
Use either a benchmark angle or **COTERMINAL ANGLE** to mark and label $-\frac{5\pi}{6}$ on the **UNIT CIRCLE** above.

• Radians Around the Unit Circle •

1) Which quadrant is $\frac{11\pi}{6}$ in?

____

Mark and label $\frac{11\pi}{6}$ on the UNIT CIRCLE.



2) Which quadrant is $-\frac{7\pi}{4}$ in?

____

Mark and label $-\frac{7\pi}{4}$ on the UNIT CIRCLE.

Mark and label positions for the angles on the UNIT CIRCLE above.

3) $\frac{\pi}{3}$

4) $-\frac{\pi}{6}$

5) $-\frac{3\pi}{2}$

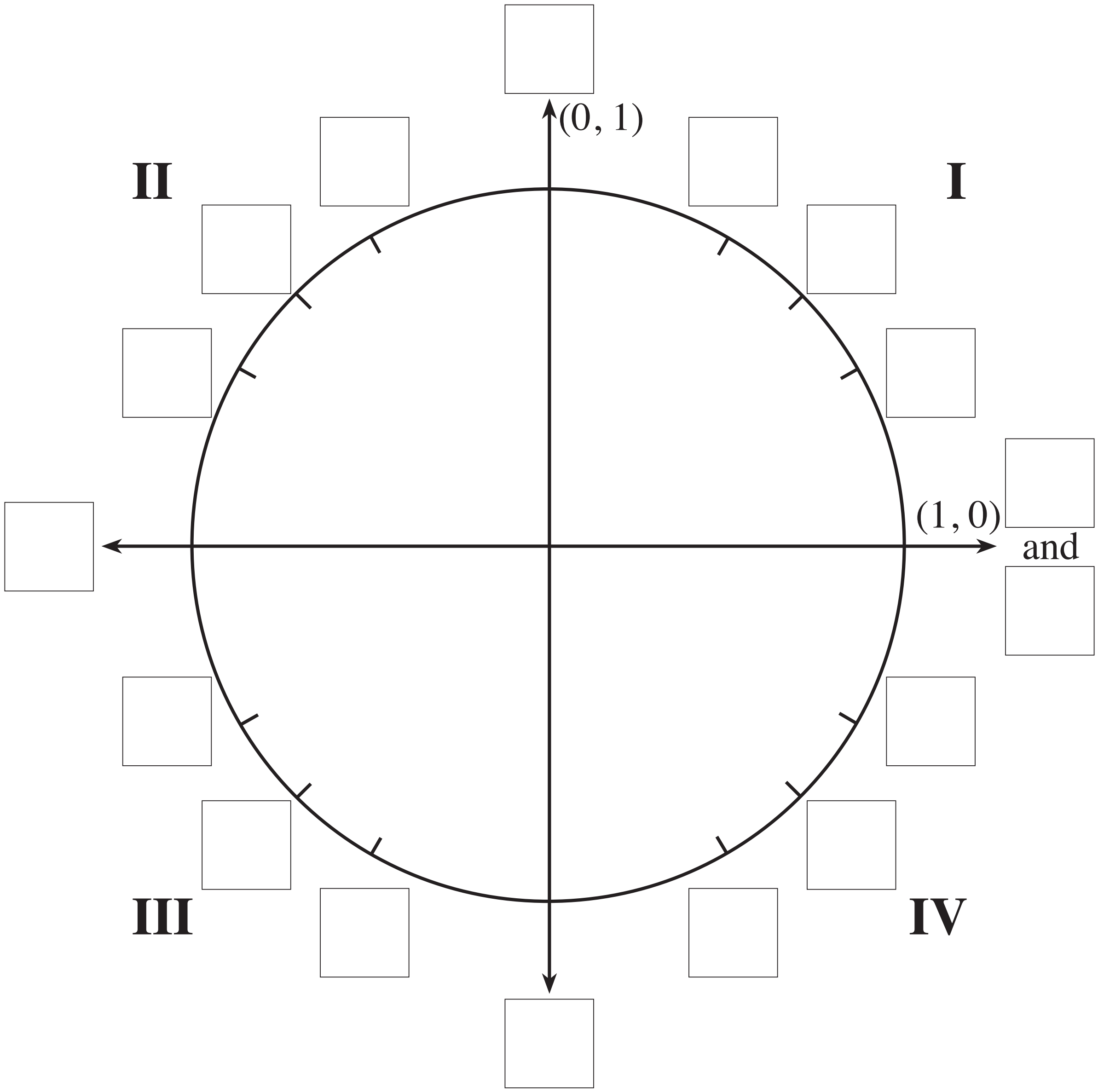
6) $\frac{3\pi}{2}$

7) $\frac{5\pi}{3}$

8) $-\frac{7\pi}{6}$

• Angles as Rotations •

Label all of the *positive* radian angles on the UNIT CIRCLE.



What strategies did you use to label the radian measurements?

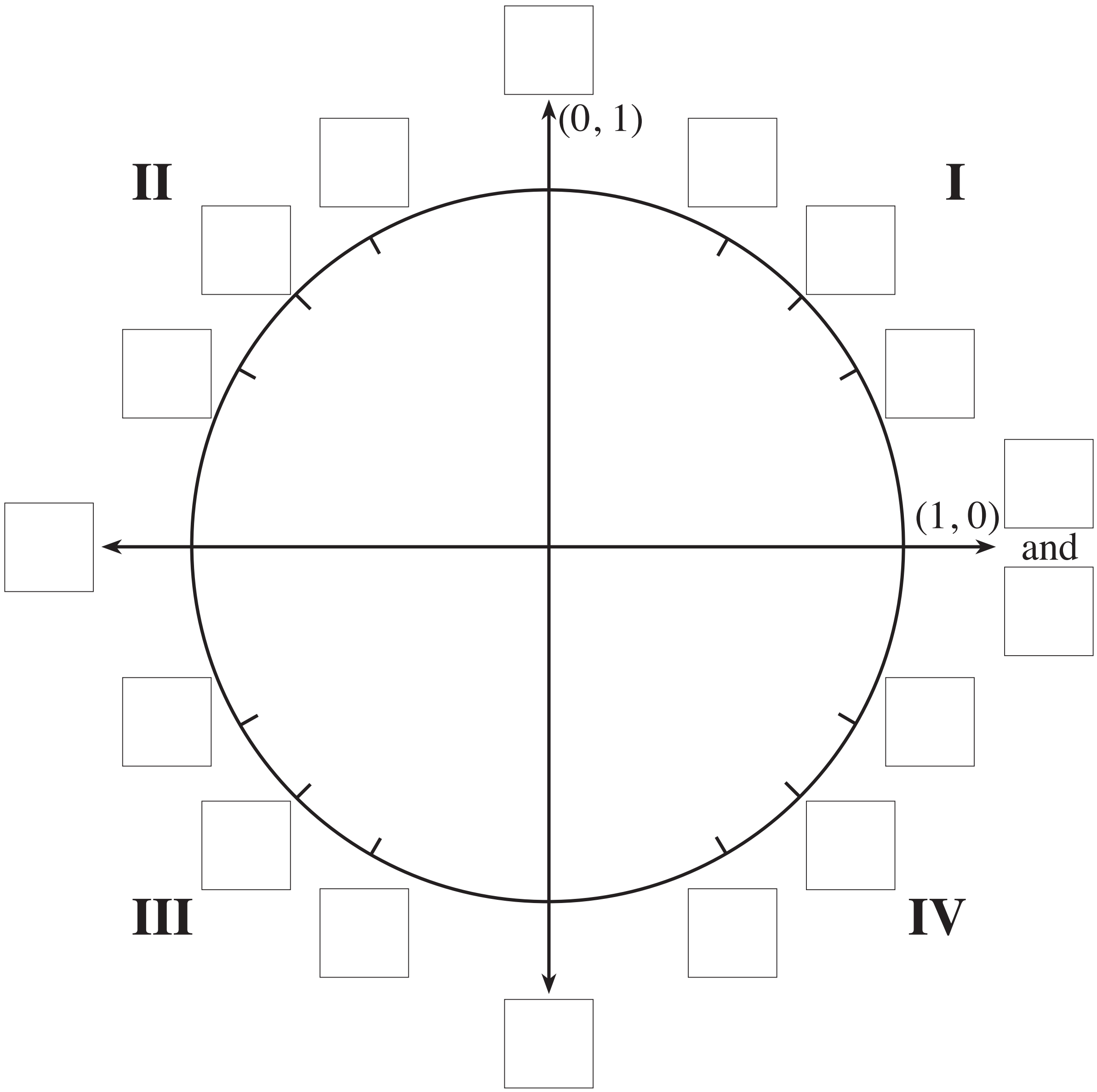
____

____

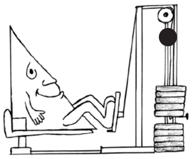
____

• Angles as Rotations •

Label all of the *negative* radian angles on the UNIT CIRCLE.



What strategies did you use to label the radian measurements?



Mastery Check: Angles as Rotations

On the UNIT CIRCLE, mark and label positions for the angles.

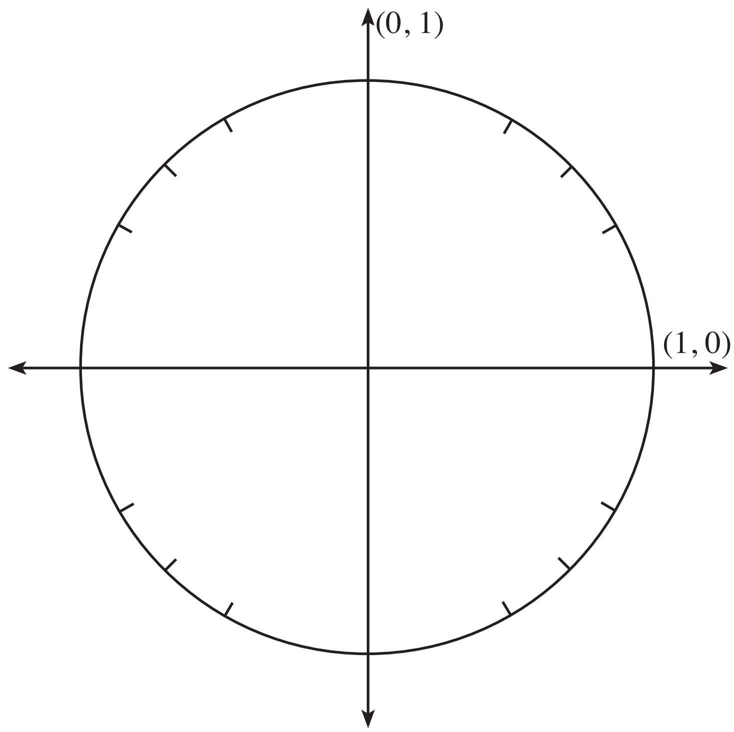
1) $\frac{\pi}{4}$

2) $-\frac{\pi}{3}$

3) $-\frac{3\pi}{2}$

4) $\frac{5\pi}{3}$

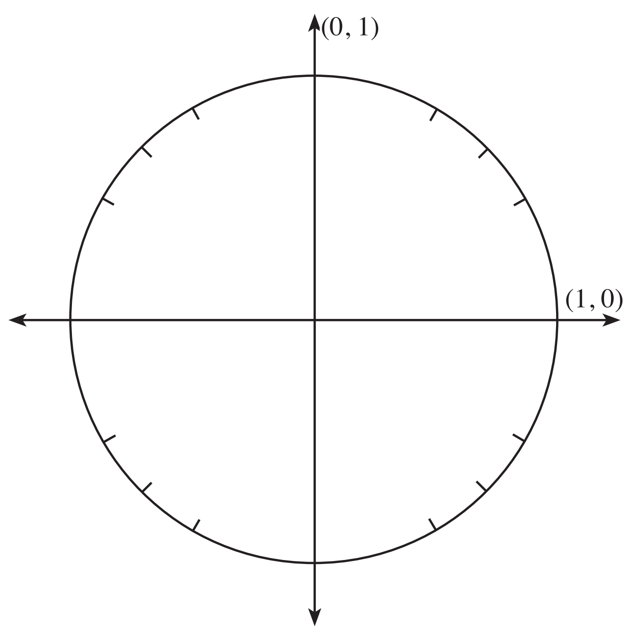
5) $-\frac{7\pi}{6}$



Challenge: On the UNIT CIRCLE, mark and label positions for the angles.

6) $\frac{7\pi}{2}$

7) $-\frac{19\pi}{6}$



• Trigonometric Ratios – Quadrant I •

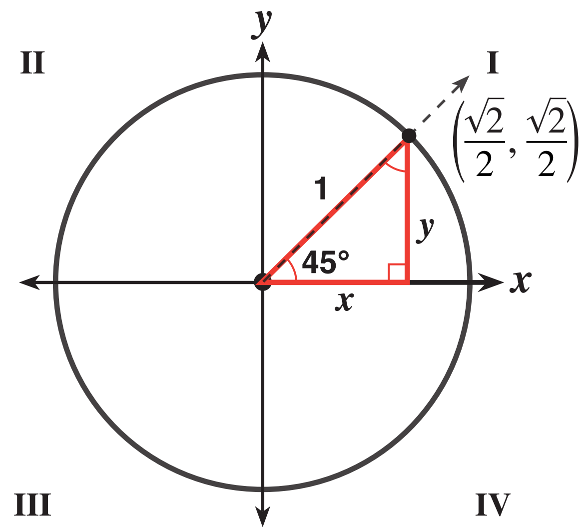
Direct Teaching

We can find trigonometric ratio values of common angle measurements from a unit circle by drawing a special right triangle in one of its quadrants. The outermost vertex of this triangle intersects the unit circle. The angle's initial side is on the positive ***x***–axis.

Example: Find cos(45°) and sin(45°) $\left(\text{or } \frac{\pi}{4}\right)$.

To visualize how to find cos(45°) and sin(45°), let's place a 45°–45°–90° triangle in Quadrant I of the unit circle. Since the unit circle has a radius of 1, the hypotenuse of the triangle has a length of 1.

We use ***x*** and ***y*** to represent the lengths of the reference triangle's legs. Using our knowledge of 45°–45°–90° triangles, we find that $x = y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.



With all side lengths of the special right triangle known, we find that:

$$\cos(45^\circ) = \frac{\text{ADJACENT}}{\text{HYPOTENUSE}} = \frac{\sqrt{2}}{2} \qquad \sin(45^\circ) = \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}} = \frac{\sqrt{2}}{2}$$

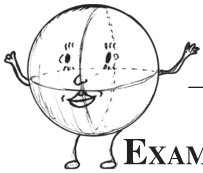
We can write the coordinate of the intersection of the triangle and unit circle as (***x***, ***y***).

$$(x, y) = (\cos(45^\circ), \sin(45^\circ)) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

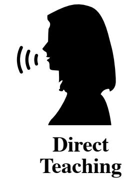
Try these: **Use the unit circle above for the following exercises.**

1) $\csc(45^\circ) = \frac{1}{\sin(45^\circ)} =$ ____________

2) $\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} =$ ____________



• Trigonometric Ratios – Quadrant I •



EXAMPLE 1: Find $\cos(30^\circ)$ and $\sin(30^\circ)$ $\left(\text{or } \frac{\pi}{6}\right)$.

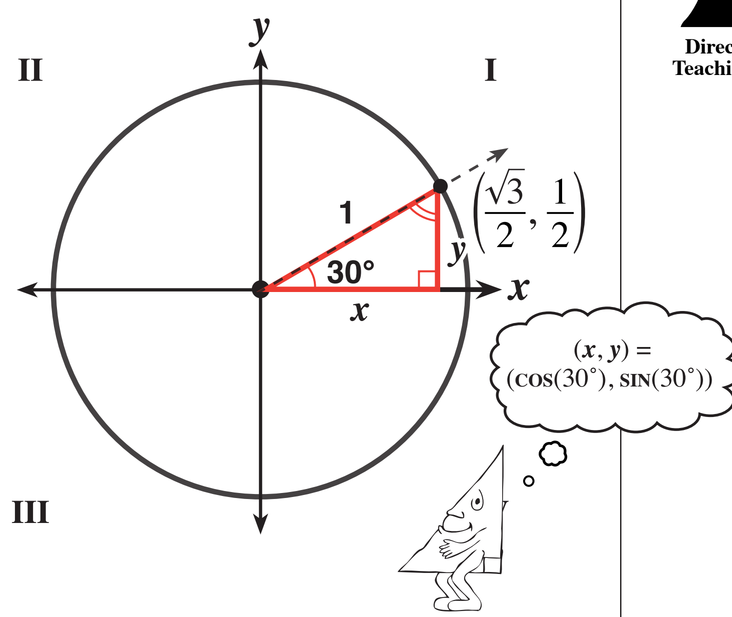
Let's place a 30°–60°–90° triangle in Quadrant I of the unit circle where the 30° angle is at the origin.

The legs of the reference triangle are represented by x and y. Using our knowledge of 30°–60°–90° triangles, we find that $x = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}$.

With all side lengths of the special right triangle known, we find that:

$$\cos(30^\circ) = \frac{\sqrt{3}}{2} \qquad \sin(30^\circ) = \frac{1}{2}$$

x–VALUE $\qquad$ y–VALUE



EXAMPLE 2: Find $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{3}\right)$ (or 60°).

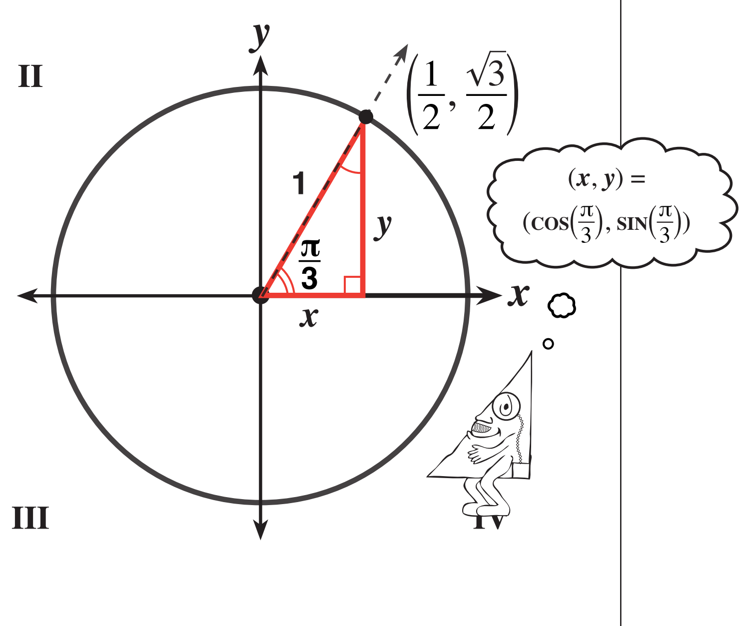
Let's place a $\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ (30°–60°–90°) triangle in Quadrant I of the unit circle where the $\frac{\pi}{3}$ angle is at the origin.

The legs of the reference triangle are represented by x and y. Using our knowledge of $\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ triangles, we find that $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$.

With all side lengths of the special right triangle known, we find that:

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

x–VALUE $\qquad$ y–VALUE



Try these: **Use the unit circles above for the following exercises.**

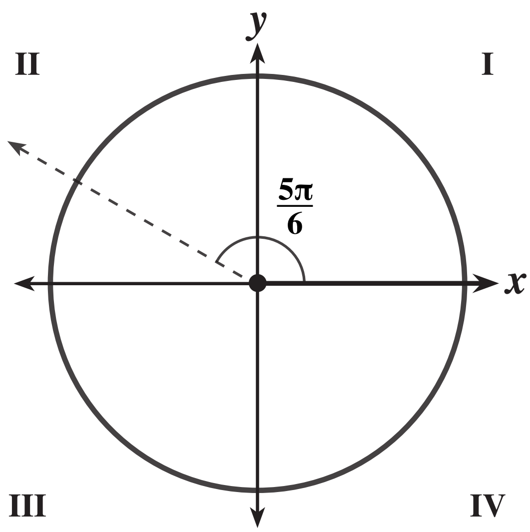
1) $\csc(30^\circ) = \frac{1}{\sin(30^\circ)} =$ __________ $\qquad$ **2)** $\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} =$ __________

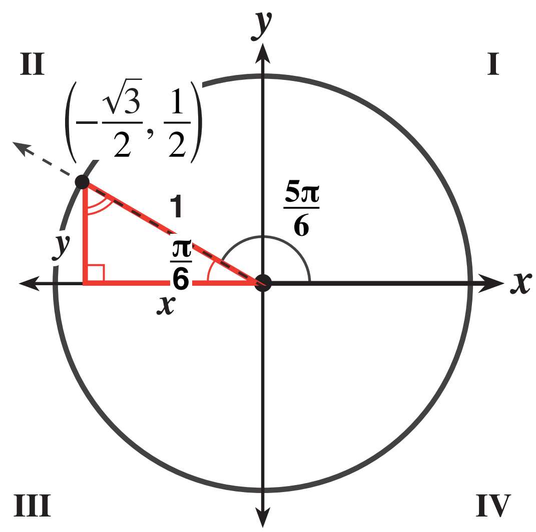
• Trigonometric Ratios – Quadrant II •

All trigonometric ratios of angles that are in Quadrant I of the unit circle are positive because x and y are both positive. When finding trigonometric ratios of angles in other quadrants, some of their values may be negative depending on the signs of x and y.

Direct Teaching

Example: Find $\cos\left(\frac{5\pi}{6}\right)$ (or 150°).





The angle that measures $\frac{5\pi}{6}$ lies in Quadrant II. Since its reference angle is $\frac{\pi}{6}$ (or 30°), we can place a $\frac{\pi}{6}-\frac{\pi}{3}-\frac{\pi}{2}$ triangle in Quadrant II of the unit circle.

The adjacent leg lies on the negative x–axis, so it can be helpful to think of its length as having a negative value. Using our knowledge of $\frac{\pi}{6}-\frac{\pi}{3}-\frac{\pi}{2}$ triangles, we know that $x = -\frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}$.

Since cosine is the x–value of the point, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

Notice how the negative x–value affects the coordinate on the unit circle above.

***Try these*:** **Use a unit circle to answer the following.**

Socratic Questioning

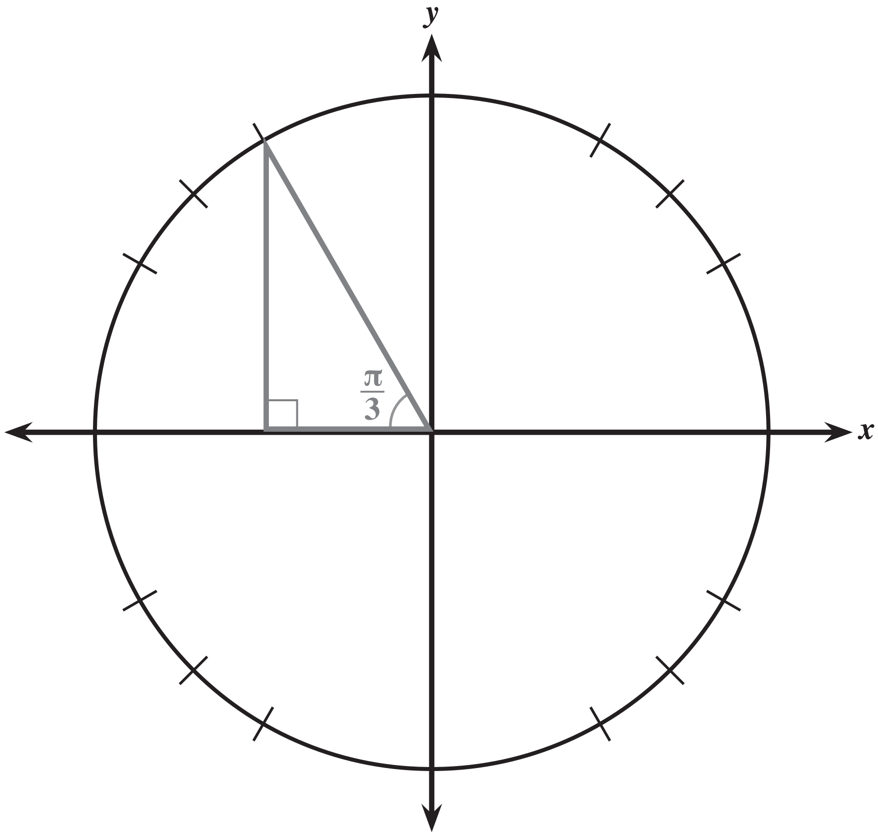
1) Find $\sin(120°)$. Is this value negative? Explain.

____

2) Which of the trigonometric ratios out of cosine, cosecant, and secant of the angle measuring 120° would be positive? Why?

____

• Trigonometric Ratios •



Use the unit circle above for the following exercises.

1) Find the reference angle and sketch the reference triangle of $\frac{2\pi}{3}$.

REFERENCE ANGLE: ____________ $\sin\left(\frac{2\pi}{3}\right) =$ ____________

2) Find the reference angle and sketch the reference triangle of 45°.

REFERENCE ANGLE: ____________ $\cos(45^\circ) =$ ____________

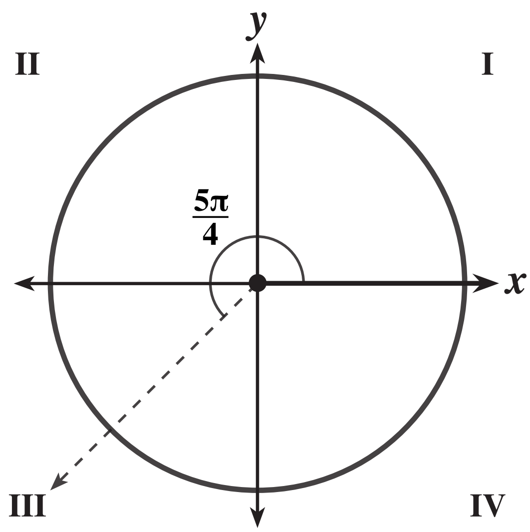
3) Find the reference angle and sketch the reference triangle of $\frac{3\pi}{4}$.

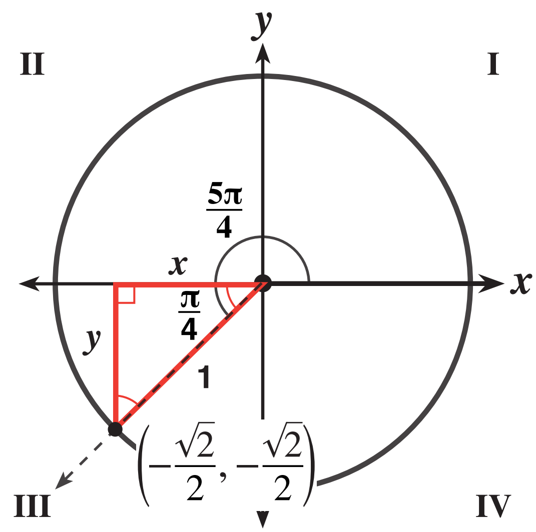
REFERENCE ANGLE: ____________ $\cos\left(\frac{3\pi}{4}\right) =$ ____________

• Trigonometric Ratios – Quadrant III •

Direct Teaching

Example: Find $\sin\left(\frac{5\pi}{4}\right)$ (or 225°).





The angle that measures $\frac{5\pi}{4}$ lies in Quadrant III. Since its reference angle is $\frac{\pi}{4}$ (or 45°), we can place a $\frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{2}$ (45°–45°–90°) triangle in Quadrant III of the unit circle where a $\frac{\pi}{4}$ angle is at the origin.

The adjacent leg lies on the negative ***x***–axis, and the opposite leg is parallel to the negative ***y***–axis. So, both legs have *negative* values. Using our knowledge of $\frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{2}$ triangles, we know that $x = -\frac{\sqrt{2}}{2}$ and $y = -\frac{\sqrt{2}}{2}$.

Since sine is the ***y***–value of the point, $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

Try these: **Use a unit circle to answer the following.**

1) cos(240°) = ___________

2) Which of the trigonometric ratios out of sine, cosine, cosecant, and secant of the angle measuring 240° would be positive? Why?

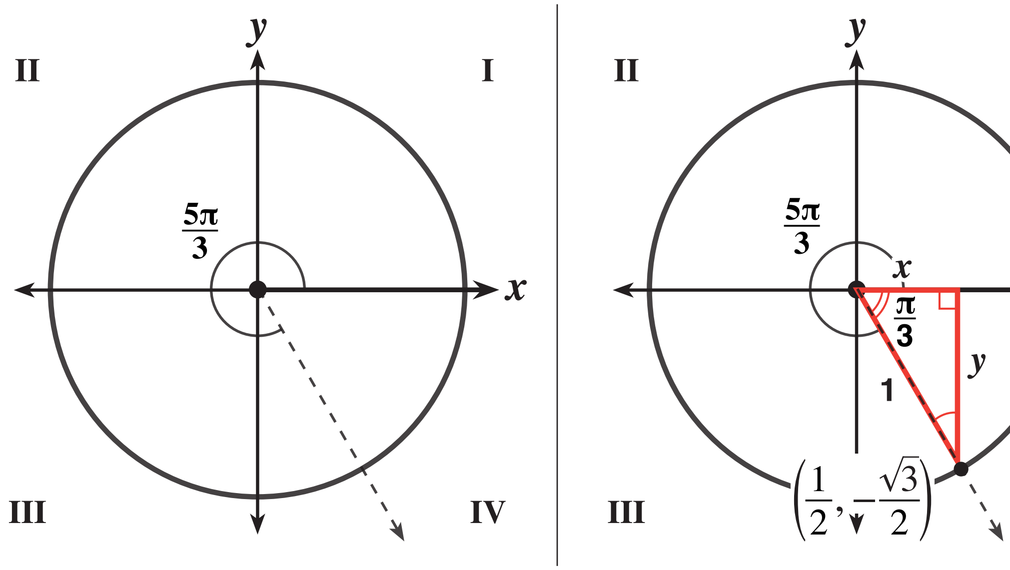
____

____

• Trigonometric Ratios – Quadrant IV •

Direct Teaching

Example: Find $\sin\left(\frac{5\pi}{3}\right)$ (or 300°).



The angle that measures $\frac{5\pi}{3}$ lies in Quadrant IV. Since its reference angle is $\frac{\pi}{3}$ (or 60°), we can place a $\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ triangle in Quadrant IV of the unit circle where the $\frac{\pi}{3}$ angle is at the origin.

The adjacent leg lies on the positive ***x***–axis, and the opposite leg is parallel to the negative ***y***–axis. So, the adjacent leg has a *positive* value, and the opposite leg has a *negative* value. Using our knowledge of $\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ triangles, we know that $x = \frac{1}{2}$ and $y = -\frac{\sqrt{3}}{2}$.

$$\text{So, } \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$$

Try these: **Use a unit circle to answer the following.**

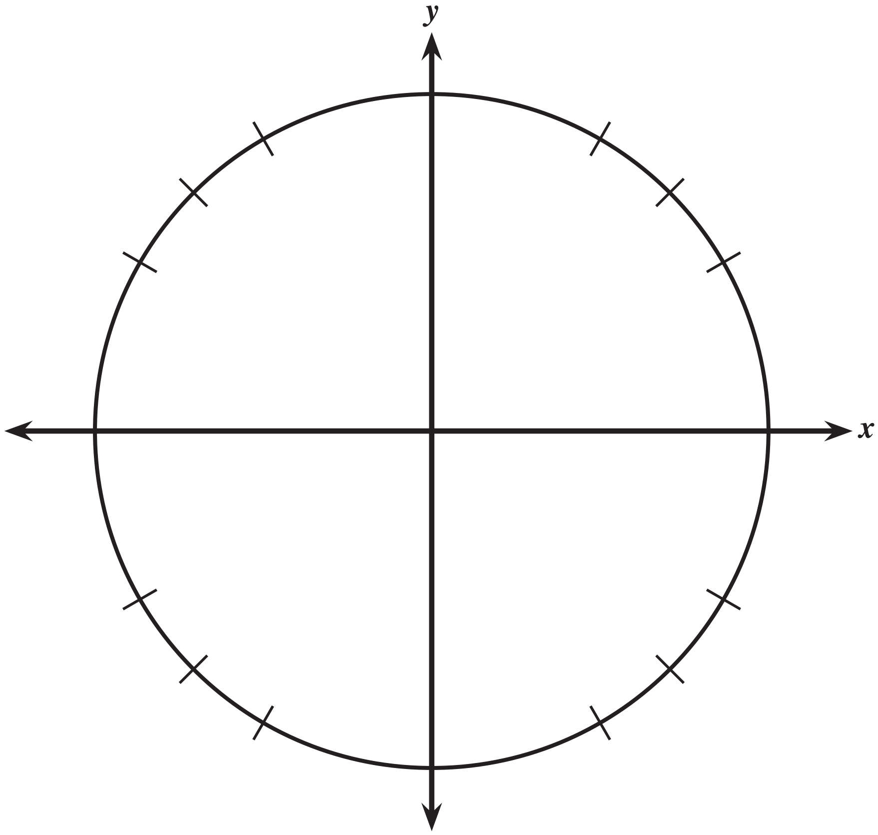
1) $\cos(330°) =$ ___________

2) Which of the trigonometric ratios out of sine, cosine, cosecant, and secant of the angle measuring 330° would be positive? Why?

____

____

• Trigonometric Ratios •



Use the unit circle above for the following exercises.

1) Find the reference angle and sketch the reference triangle of $\frac{7\pi}{6}$.

REFERENCE ANGLE: ____________ $\cos\left(\frac{7\pi}{6}\right) =$ ____________

2) Find the reference angle and sketch the reference triangle of 120°.

REFERENCE ANGLE: ____________ $\sin(120^\circ) =$ ____________

3) Find the reference angle and sketch the reference triangle of $\frac{7\pi}{4}$.

REFERENCE ANGLE: ____________ $\cos\left(\frac{7\pi}{4}\right) =$ ____________

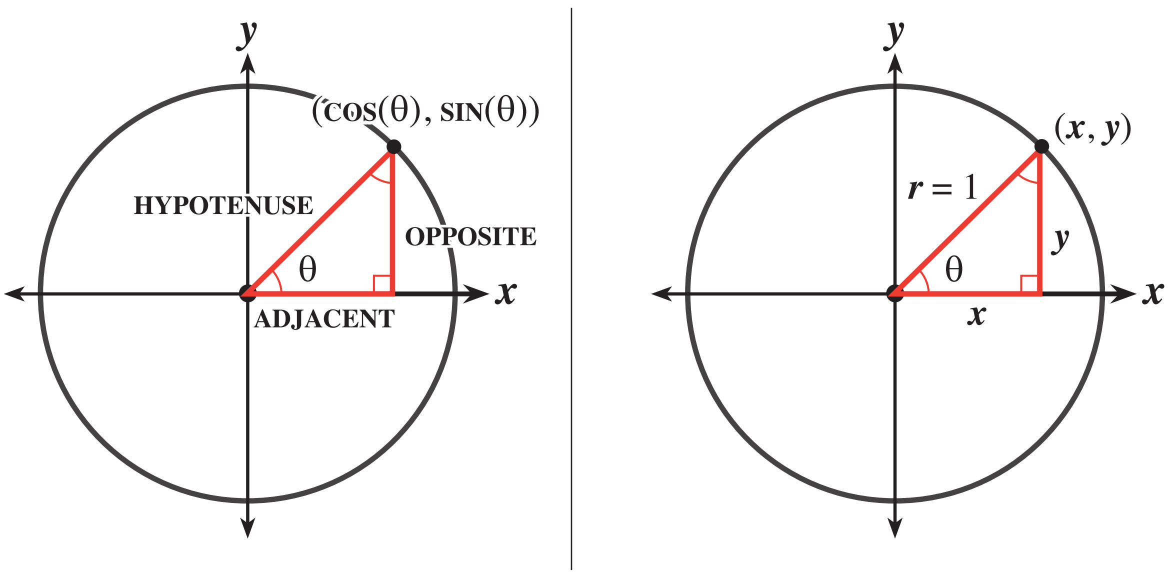
• Right Triangles and the Unit Circle •

Let's take a closer look at the connection between the sides of a right triangle and coordinates on the unit circle.

As shown below for θ in Quadrant I, the right triangle drawn on the unit circle has leg lengths represented by x and y. The length of its hypotenuse is equal to the radius of the unit circle, r, which is always equal to 1.

Since the hypotenuse of a right triangle drawn on a unit circle is always equal to 1, we can say that for cos(θ) and sin(θ):

$$\cos(\theta) = \frac{\text{ADJACENT}}{\text{HYPOTENUSE}} = \frac{x}{r} = \frac{x}{1} = x \qquad \sin(\theta) = \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}} = \frac{y}{r} = \frac{y}{1} = y$$



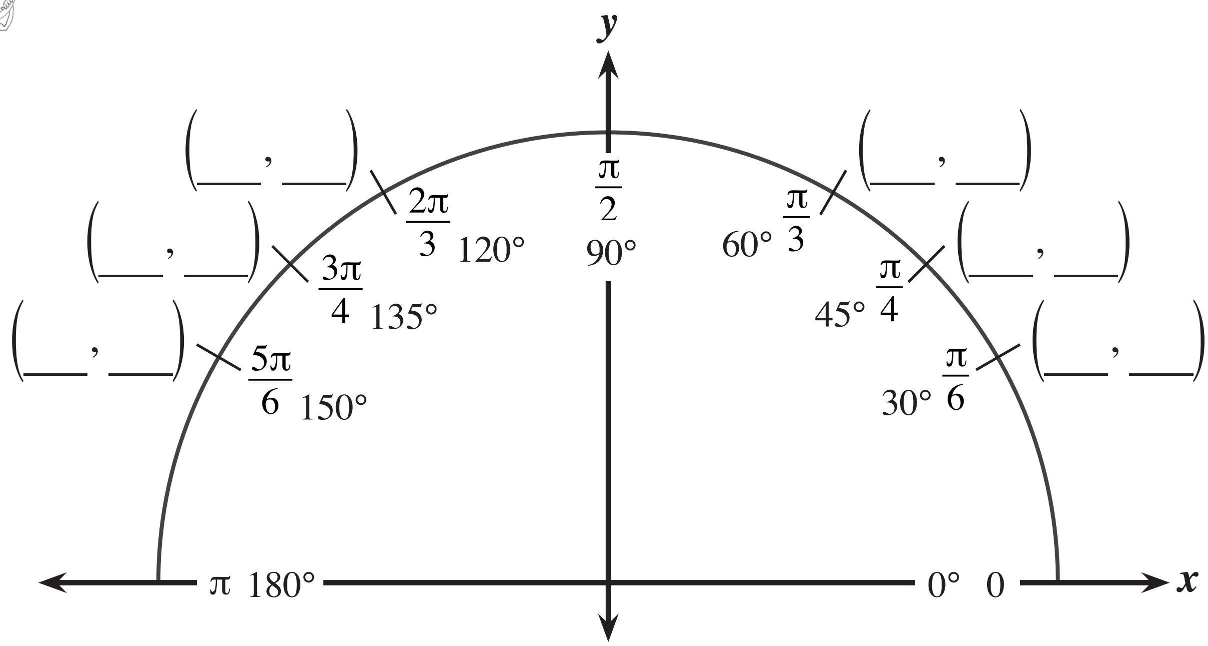
So, the coordinate $(x, y) = (\cos(\theta), \sin(\theta))$.

Using x, y, and r, fill in the blanks for the rest of the trigonometric ratios.

1) $\csc(\theta) = \frac{r}{y} = \frac{1}{y}$

2) $\sec(\theta) = \frac{\quad}{\quad} = \frac{\quad}{\quad}$

• Unit Circle – Quadrant I and Quadrant II •



Use Quadrant I and Quadrant II of the unit circle to complete the following exercises.

1) Plot and label the coordinate of each common angle shown in Quadrant I. Sketch reference triangles if needed.

2) Use the coordinate of 60˚ to plot and label the coordinate of 120˚. What do you notice about the reflectional symmetry about the ***y***–axis?

____

____

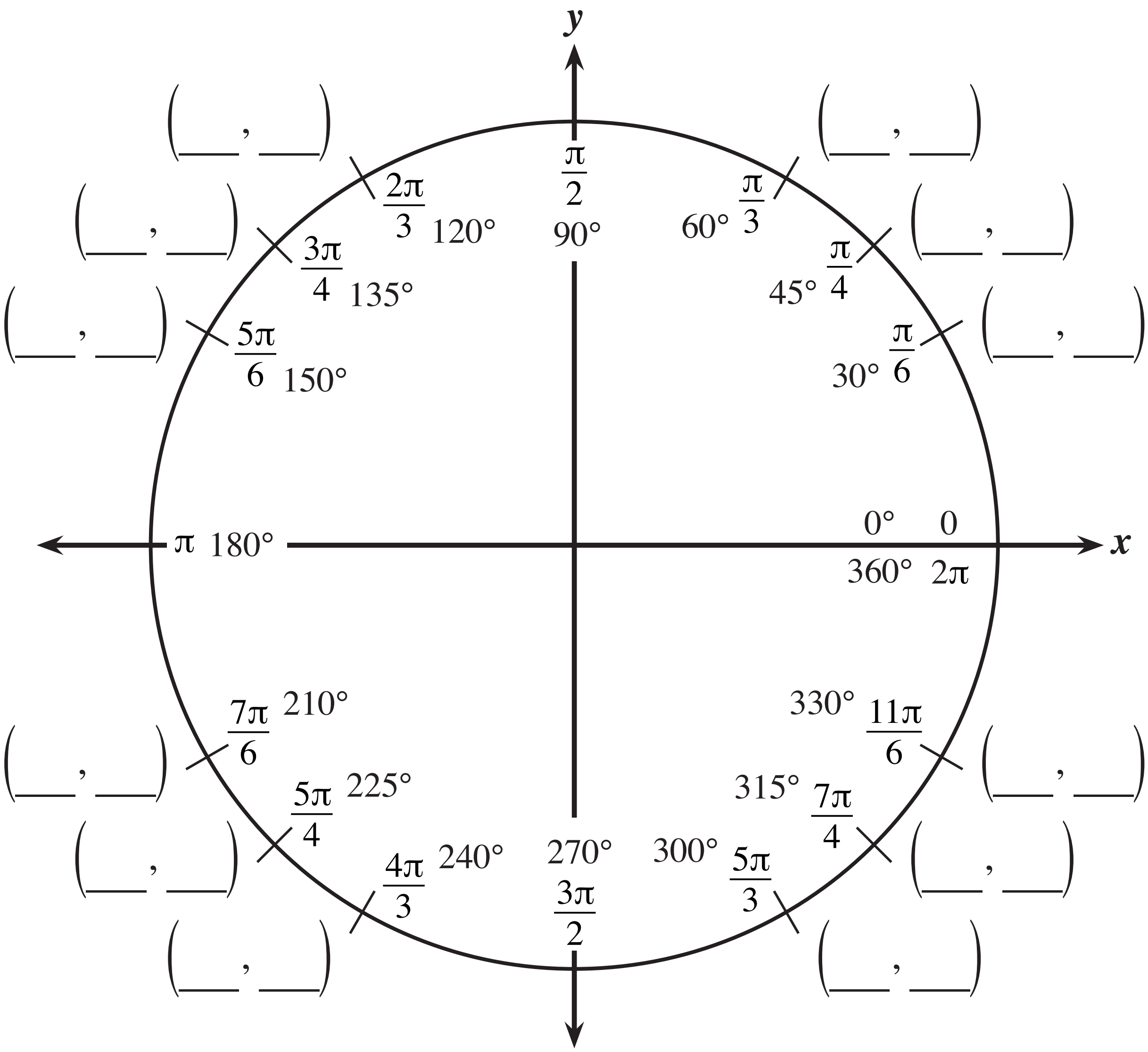
3) Using your plotted coordinates in Quadrant I and properties of symmetry, plot and label the coordinates of 135˚ and 150˚.

4) $\cos(30^\circ) = x =$ ____________

5) $\sin\left(\frac{5\pi}{6}\right) = y =$ ____________

• Unit Circle – All Quadrants •

Once we know the trigonometric ratio values for the common angle measurements in Quadrant I, we can figure out the values for the common angle measurements in the other quadrants.



1) Label the angle and unit circle intersections as coordinates on the unit circle above. **Hint**: Start with Quadrant I, and think about the signs of the axes in each quadrant.

2) $\sin(240^\circ) = y =$ ___________

3) $\cos\left(\frac{11\pi}{6}\right) = x =$ ___________

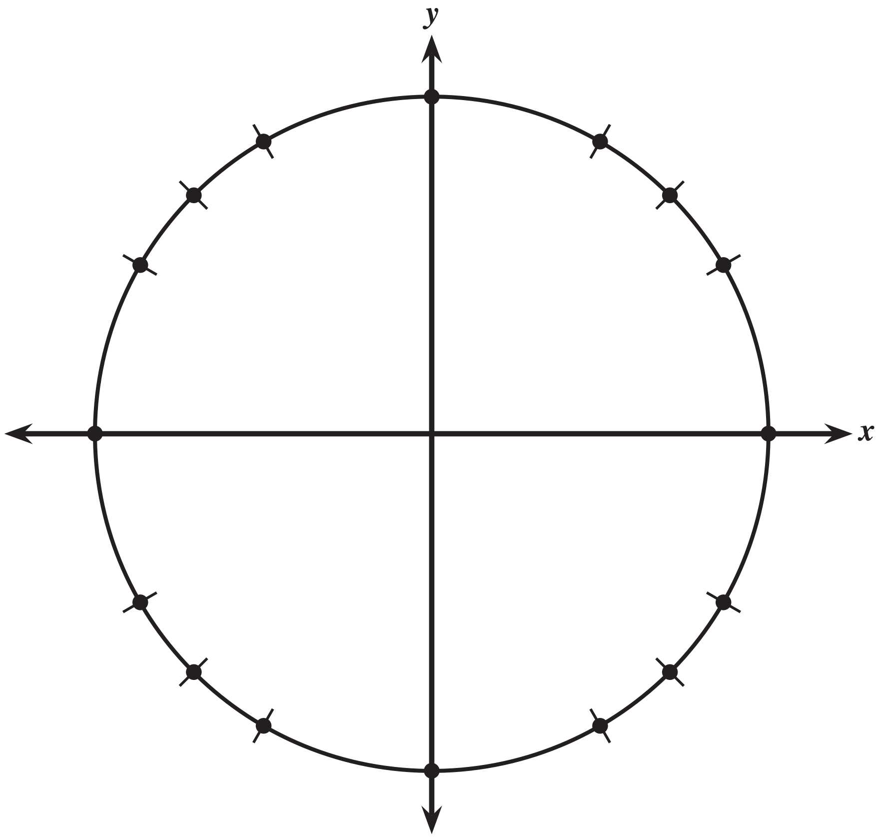
4) $\csc\left(\frac{3\pi}{4}\right) = \frac{1}{y} =$ ___________

5) $\sec(45^\circ) = \frac{1}{x} =$ ___________

• Unit Circle •

$\text{SIN}(\theta) = y$ $\text{COS}(\theta) = x$

$\text{CSC}(\theta) = \frac{1}{y}$ $\text{SEC}(\theta) = \frac{1}{x}$



1) Write each common angle measurement in degrees and radians on the unit circle above.

2) Label the angle and unit circle intersections as coordinates on the unit circle above.

3) $\text{SEC}(135^\circ) =$ ___________

4) $\text{COS}\left(\frac{11\pi}{6}\right) =$ ___________

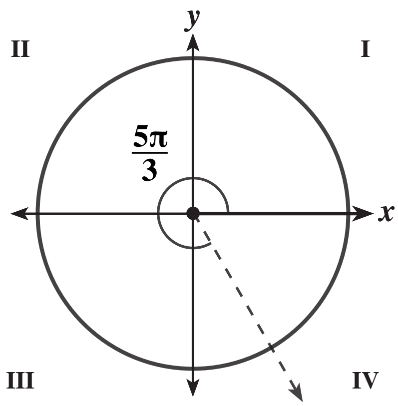
5) $\text{SIN}\left(\frac{3\pi}{4}\right) =$ ___________

6) $\text{CSC}(300^\circ) =$ ___________

• Finding Tangent •

In a right triangle, $\text{TAN}(\theta) = \frac{\text{OPPOSITE}}{\text{ADJACENT}}$. With reference triangles, $\text{OPPOSITE} = \text{SIN}(\theta)$ (the y–value), and $\text{ADJACENT} = \text{COS}(\theta)$ (the x–value). So, $\text{TAN}(\theta) = \frac{\text{OPPOSITE}}{\text{ADJACENT}} = \frac{\text{SIN}(\theta)}{\text{COS}(\theta)} = \frac{y}{x}$.

Example: Find $\text{TAN}\left(\frac{5\pi}{3}\right)$.



$\frac{5\pi}{3}$ lies in Quadrant IV, so we know that $\text{COS}\left(\frac{5\pi}{3}\right)$ is positive and $\text{SIN}\left(\frac{5\pi}{3}\right)$ is negative. The reference angle of $\frac{5\pi}{3}$ is $\frac{\pi}{3}$, which we can use to find the sine and cosine values.

$$(x, y) = \left(\text{COS}\left(\tfrac{5\pi}{3}\right), \text{SIN}\left(\tfrac{5\pi}{3}\right)\right) = \left(\text{COS}\left(\tfrac{\pi}{3}\right), {}^{-}\text{SIN}\left(\tfrac{\pi}{3}\right)\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Now, we can find the tangent value.

$$\text{TAN}\left(\tfrac{5\pi}{3}\right) = \frac{y}{x} = \frac{\text{SIN}\left(\frac{5\pi}{3}\right)}{\text{COS}\left(\frac{5\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = {}^{-}\sqrt{3}$$

***Try these*:** **Find each trigonometric ratio.**

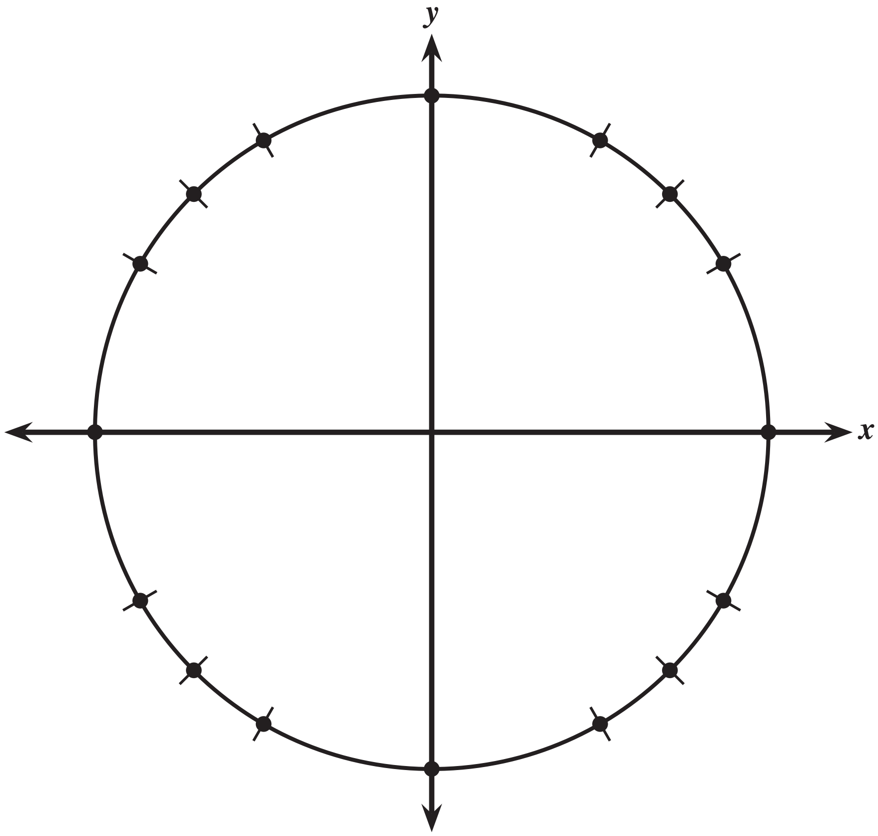
1) Find $\text{TAN}(315^\circ)$. REFERENCE ANGLE: ___________

$(x, y) = ($_______, _______$)$ $\text{TAN}(315^\circ) =$ ___________

2) Find $\text{COT}\left(\frac{7\pi}{6}\right)$. REFERENCE ANGLE: ___________

$(x, y) = ($_______, _______$)$ $\text{COT}\left(\frac{7\pi}{6}\right) =$ ___________

• Sine, Cosine, and Tangent •



1) Write each common angle measurement in degrees and radians on the unit circle above.

2) Label the angle and unit circle intersections as coordinates on the unit circle above.

3) Fill in the table below.

	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\frac{7\pi}{6}$	$\frac{7\pi}{4}$
(x, y)					
SINE					
COSINE					
TANGENT					

Socratic Questioning

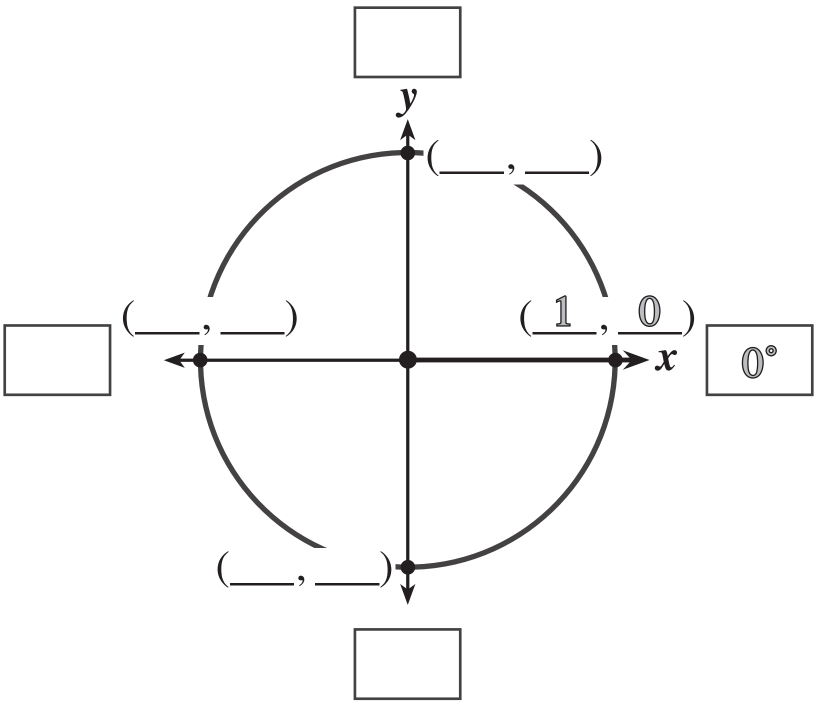
• Quadrantal Angles •

Direct Teaching

A **QUADRANTAL ANGLE** is an angle in standard position on a coordinate plane whose terminal side is on one of the axes. There are four coordinates on a unit circle that represent **QUADRANTAL ANGLES**.

Try these:

EXAMPLE 1: Fill in the plotted coordinates and label the **QUADRANTAL ANGLES** in degrees on the unit circle below.



EXAMPLE 2: Use the coordinates labeled and plotted from **EXAMPLE 1** to find the following trigonometric ratios: CSC(180°), TAN(180°), and SEC(360°).

$$\csc(180^\circ) = \frac{1}{y} = \frac{1}{0} = \underline{\varnothing}$$

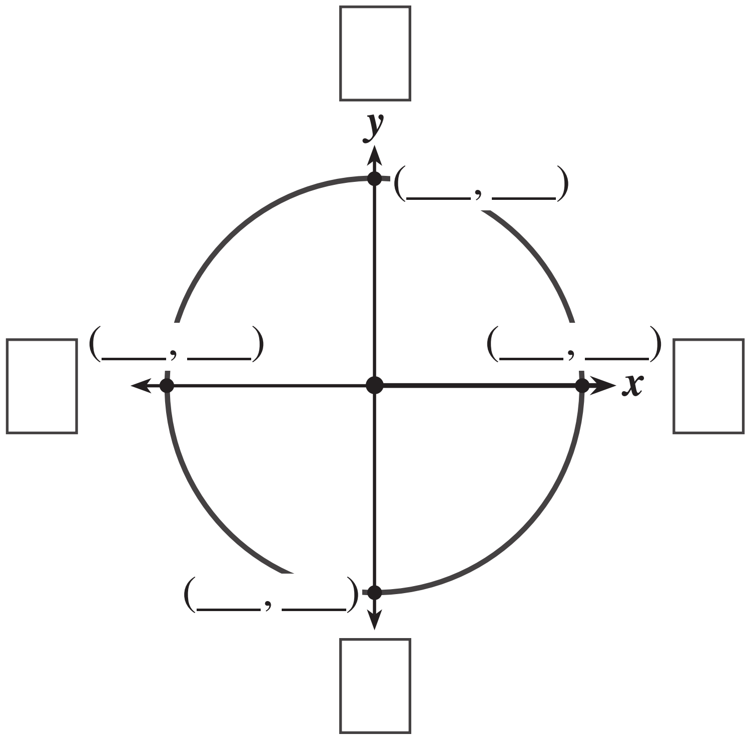
Since a fraction is undefined when its denominator is equal to 0, we use the symbol Ø to denote that $\csc(\theta)$ is undefined at $\theta = 180^\circ$.

$$\tan(180^\circ) = \frac{y}{x} = \frac{\quad}{\quad} = \underline{\qquad}$$

$$\sec(360^\circ) = \frac{\quad}{\quad} = \frac{\quad}{\quad} = \underline{\qquad}$$

360° has the same coordinate as 0°. So, we also use (1,0) to find the trigonometric ratios for 360°.

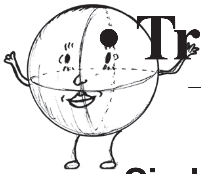
• Trigonometric Ratios – Quadrantal Angles •



1) Fill in the plotted coordinates and label the quadrantal angles in radians on the unit circle above.

2) Fill in the table below.

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
(x, y)					
SINE					
COSINE					
TANGENT					
COSECANT					
SECANT					
COTANGENT					



• Trigonometric Ratios – Positive or Negative? • ☆

Circle the correct sign for each trigonometric ratio value within each quadrant.

1)

y

II

SINE: + / –
COSINE: + / –
TANGENT: + / –
COSECANT: + / –
SECANT: + / –
COTANGENT: + / –

I

SINE: + / –
COSINE: + / –
TANGENT: + / –
COSECANT: + / –
SECANT: + / –
COTANGENT: + / –

x

III

SINE: + / –
COSINE: + / –
TANGENT: + / –
COSECANT: + / –
SECANT: + / –
COTANGENT: + / –

IV

SINE: + / –
COSINE: + / –
TANGENT: + / –
COSECANT: + / –
SECANT: + / –
COTANGENT: + / –

---

2) $\cos(45^\circ)$: + / –

3) $\csc\left(\frac{11\pi}{6}\right)$: + / –

4) $\sec\left(\frac{2\pi}{3}\right)$: + / –

5) $\tan(210^\circ)$: + / –

• Trigonometric Ratios of an Angle •

Trigonometric Ratios

$\cos(\theta) = x$	$\sin(\theta) = y$	$\tan(\theta) = \frac{y}{x}$
$\sec(\theta) = \frac{1}{x}$	$\csc(\theta) = \frac{1}{y}$	$\cot(\theta) = \frac{x}{y}$

1) Find $\csc\left(\frac{5\pi}{4}\right)$.

$(x, y) = (\_\_\_\_\_\_\_, \_\_\_\_\_\_\_)$ $\csc\left(\frac{5\pi}{4}\right) = \frac{1}{y} = \_\_\_\_\_\_\_\_\_\_\_$

2) Find $\cot(30^\circ)$.

$(x, y) = (\_\_\_\_\_\_\_, \_\_\_\_\_\_\_)$ $\cot(30^\circ) = \frac{x}{y} = \_\_\_\_\_\_\_\_\_\_\_$

3) $\tan\left(\frac{4\pi}{3}\right) = \_\_\_\_\_\_\_\_\_\_\_$

4) $\cos(150^\circ) = \_\_\_\_\_\_\_\_\_\_\_$

5) $\sin(30^\circ) = \_\_\_\_\_\_\_\_\_\_\_$

6) $\tan\left(\frac{7\pi}{4}\right) = \_\_\_\_\_\_\_\_\_\_\_$

7) $\sec(270^\circ) = \_\_\_\_\_\_\_\_\_\_\_$

8) $\csc\left(\frac{2\pi}{3}\right) = \_\_\_\_\_\_\_\_\_\_\_$

• Trigonometric Ratios of an Angle •

Fill in the tables below.

1)

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
(x, y)					
SINE					
COSINE					
TANGENT					
COSECANT					
SECANT					
COTANGENT					

2)

	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
(x, y)				
SINE				
COSINE				
TANGENT				
COSECANT				
SECANT				
COTANGENT				

• Trigonometric Ratios of an Angle •

Fill in the tables below.

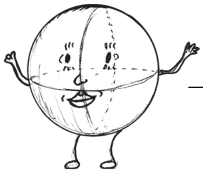
1)

	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
(x, y)				
SINE				
COSINE				
TANGENT				
COSECANT				
SECANT				
COTANGENT				

2)

	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(x, y)				
SINE				
COSINE				
TANGENT				
COSECANT				
SECANT				
COTANGENT				

Bypass



• Trigonometric Ratios of an Angle •

1) $\sin\left(\frac{\pi}{2}\right) =$ ____________

2) $\cos(240^\circ) =$ ____________

3) $\csc\left(\frac{5\pi}{6}\right) =$ ____________

4) $\cot\left(\frac{5\pi}{3}\right) =$ ____________

5) $\sec(\pi) =$ ____________

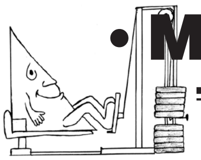
6) $\cos\left(\frac{7\pi}{6}\right) =$ ____________

7) $\tan(0^\circ) =$ ____________

8) $\sin\left(\frac{3\pi}{4}\right) =$ ____________

9) $\cot\left(\frac{11\pi}{6}\right) =$ ____________

10) $\csc(45^\circ) =$ ____________



• Mastery Check: Trig Ratios of an Angle • ☆

1) $\cos(30^\circ) =$ ____________

2) $\sin\left(\frac{7\pi}{4}\right) =$ ____________

3) $\tan(90^\circ) =$ ____________

4) $\csc\left(\frac{4\pi}{3}\right) =$ ____________

5) $\cot\left(\frac{3\pi}{2}\right) =$ ____________

6) $\tan(300^\circ) =$ ____________

7) $\sec(180^\circ) =$ ____________

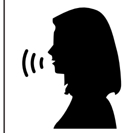
8) $\cos\left(\frac{5\pi}{6}\right) =$ ____________

Challenge:

9) $\sin(^{-}30^\circ) =$ ____________

10) $\cos\left(-\frac{7\pi}{2}\right) =$ ____________

• Sketching sin(x) •

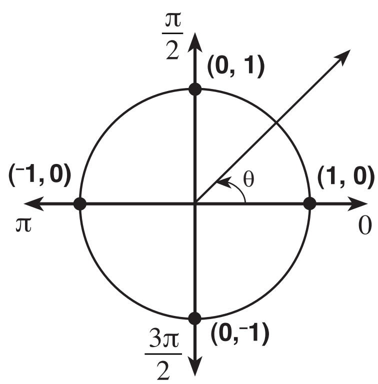


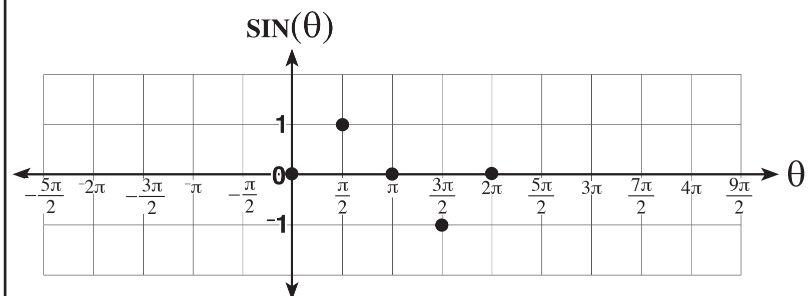
There will be five **KEY POINTS** that lie on the graph of a sine function: three x–intercepts, a minimum, and a maximum.

Try this:

EXAMPLE: Sketch the function, $f(x) = \sin(x)$, by finding key points.

The **KEY POINTS** define a **PERIOD** of the graph and are associated to points on the unit circle. The key points divide the period into four evenly spaced sub-intervals. The x–axis of the graph represents the angle rotated in radians, and the y–axis represents the sine value from the unit circle. Each point on the graph is represented by $(\theta, \sin(\theta))$.

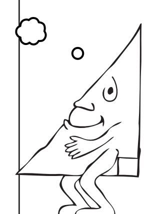




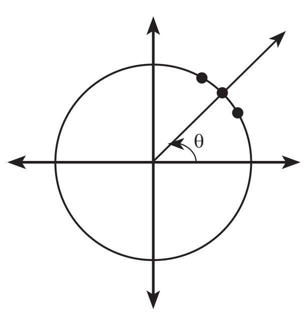
The function $f(x) = \sin(x)$ is periodic, and in this case the **PERIOD** is 2π. Fill in the missing values below for the **PERIOD** before and after the given one.

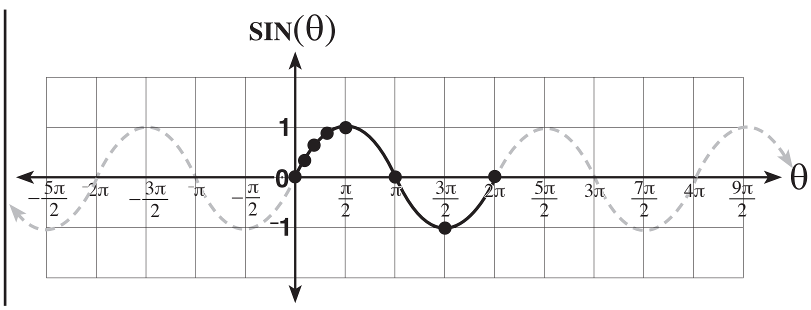
$f(x) = \sin(x)$													
x	$^{-}2\pi$	$-\frac{3\pi}{2}$	$^{-}\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
y					0	1	0	$^{-}1$	0				

Look for a pattern forming between the values given in the table.



To connect the **KEY POINTS**, we can look at the values of sine for the special angles in each quadrant. The points $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$, and $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ are included below. These give the function its shape, and this continues due to its periodic nature.





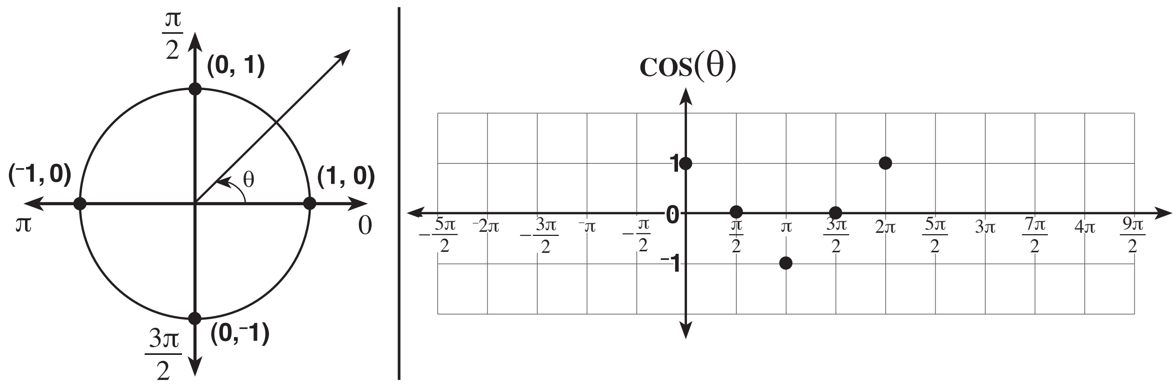
Here the domain of the sine function is $(^{-}\infty, \infty)$, and the range is $[^{-}1, 1]$.

• Sketching cos(*x*) •

Let's take a look at the key points of a cosine function. There will be five key points that lie on the curve: two *x*–intercepts, two maximums, and a minimum.

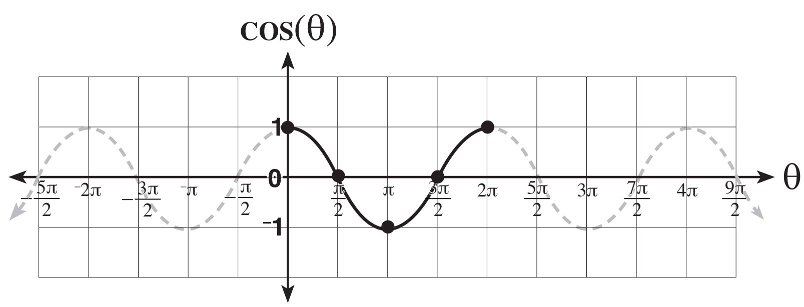
EXAMPLE: Sketch the function, $f(x) = \cos(x)$, by finding key points.

Each key point defines a section of the graph and is associated to a point on the unit circle. The *x*–axis of the graph represents an angle in radians, and the *y*–axis represents the cosine value from the unit circle. Each point on the graph is represented by $(\theta, \cos(\theta))$.



The function $f(x) = \cos(x)$ is periodic, and in this case the **PERIOD** is 2π.

$f(x) = \cos(x)$													
x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
y	1	0	⁻1	0	1	0	⁻1	0	1	0	⁻1	0	1

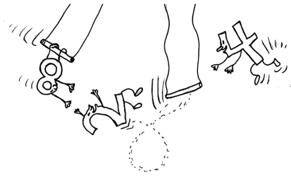


Here the domain of the cosine function is $(-\infty, \infty)$, and the range is $[-1, 1]$.

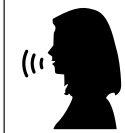
Try this:

1) State a similarity between the graphs of the sine and cosine functions.

____



• Amplitude •



Direct Teaching

$$f(x) = a\text{SIN}x$$

Trigonometric functions can be sketched by transforming the key points. Let's first take a look at the transformations that affect the AMPLITUDE, $|a|$, of a function, or half the vertical distance from its minimum to its maximum. The AMPLITUDE is always a positive value. As these functions repeat periodically, we will focus on just one period with the five key points.

Example: Sketch the function $f(x) = 2\text{SIN}(x)$.

When a trigonometric function is multiplied by a coefficient, its AMPLITUDE is affected. Here, the value of $|a| = 2$. Let's look at the points that make up the function $f(x) = 2\text{SIN}(x)$ and compare them to the points of the function $f(x) = \text{SIN}(x)$.

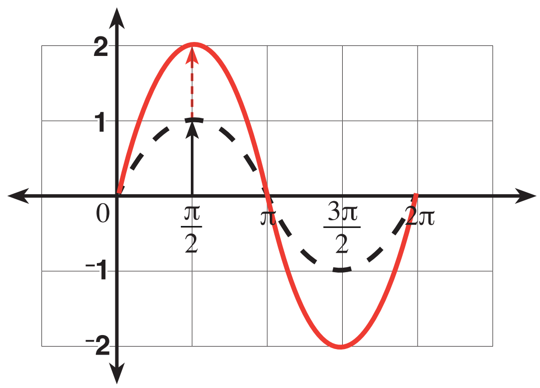
$f(x) = \text{SIN}(x)$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	1	0	$^-1$	0

×2

$f(x) = 2\text{SIN}(x)$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	2	0	$^-2$	0



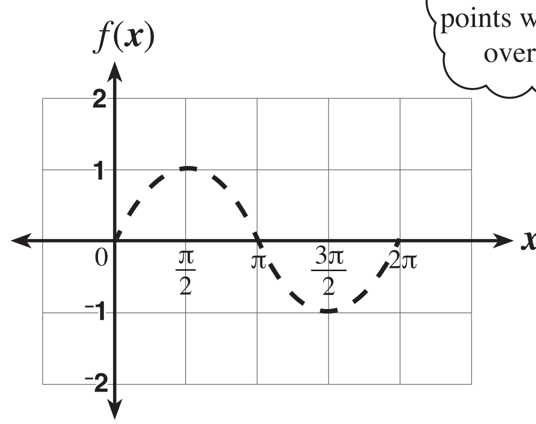
The x–values are the same in both cases, but the y–values are multiplied by $|a|$, or 2 in this case. Here, the points are stretched vertically. The points on the curve will stretch vertically when $|a| > 1$. The points will compress vertically when $|a| < 1$.

***Try this*: Sketch the function.**

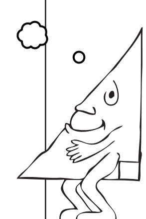
1) $f(x) = -\frac{1}{2}\text{SIN}(x)$

$f(x) = -\frac{1}{2}\text{SIN}(x)$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y					



When there is a negative sign in front of the function, the points will be reflected over the x–axis.



• Amplitude •

☆

Sketch the functions. State their amplitudes and list key points.

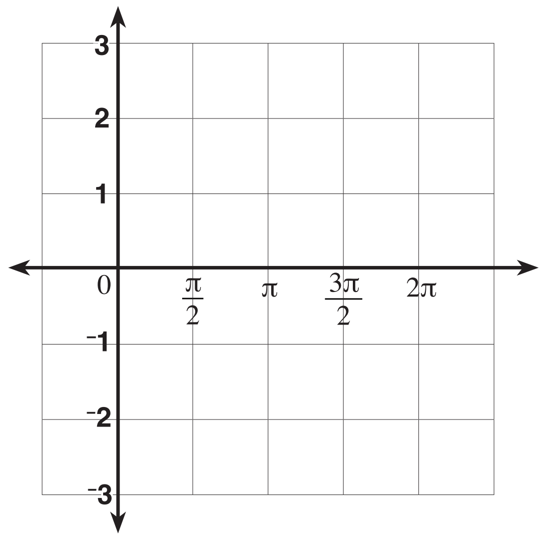
1) $f(x) = 3\cos(x)$

AMPLITUDE = $|a|$ = ____________

$f(x) = \cos(x)$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	$^{-}1$	0	1

$\times 3$

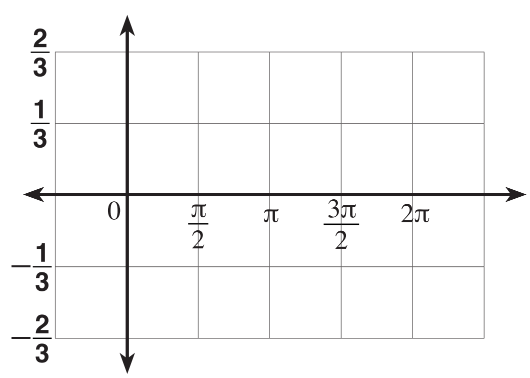
$f(x) = 3\cos(x)$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y					



2) $f(x) = \frac{1}{3}\cos(x)$

AMPLITUDE = $|a|$ = ____________

$f(x) = \frac{1}{3}\cos(x)$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y					



• Period •

$f(x) = \text{SIN}(bx)$

Next, let's take a look at the transformations that affect the **PERIOD** of a function

EXAMPLE: Sketch the function $f(x) = \cos(2x)$.

When the argument in a trigonometric function is multiplied by a coefficient, the **PERIOD** is affected. The period is defined by $\frac{2\pi}{b}$. Here, the value of $b = 2$. Let's look at the points that make up the function $f(x) = \cos(2x)$ and compare them to the points of the original cosine function. The period of $f(x) = \cos(x)$ is $\frac{2\pi}{1}$, or 2π. We can determine the **PERIOD** of $f(x) = \cos(2x)$ to be $\frac{2\pi}{2}$, or π. This is how often the function repeats.

From the starting point, 0, to the ending point of a period, there are 4 intervals that are evenly spaced horizontally. We can divide the period by 4 to find the distance between the intervals, $\frac{\pi}{4}$.

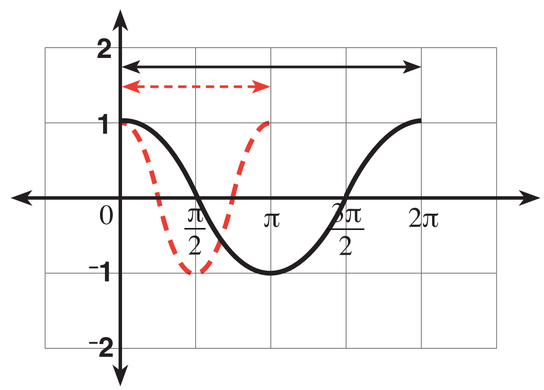
$f(x) = \cos(x)$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	$^-1$	0	1

$+\frac{\pi}{4}$ $+\frac{\pi}{4}$ $+\frac{\pi}{4}$ $+\frac{\pi}{4}$

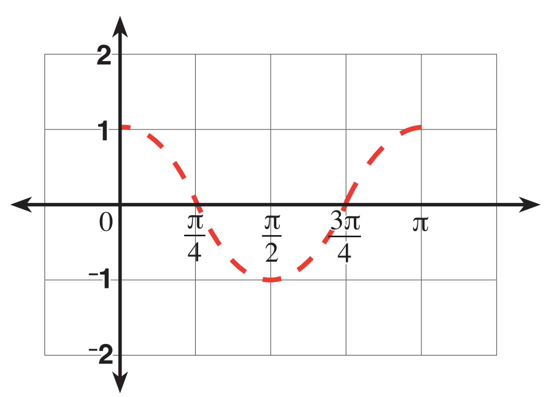
$f(x) = \cos(2x)$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	1	0	$^-1$	0	1



The y–values are the same in both cases. Here, the points are compressed horizontally. The points on the curve will compress horizontally when $b > 1$. The points will stretch horizontally when $b < 1$.

When graphing trigonometric functions, we may need to adjust the units on the x–axis to take in to account the new x–values. Here is $f(x) = \cos(2x)$ with new x–values labeled.



• Period •

Sketch the functions. State the period and list key points.

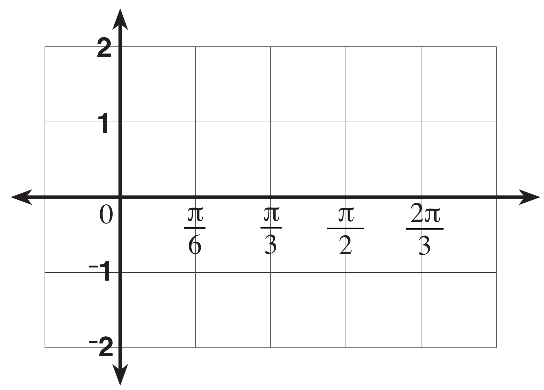
1) $f(x) = \cos(3x)$

$\text{PERIOD} = \dfrac{2\pi}{b} = $ __________

$\text{INTERVAL SIZE} = \dfrac{\text{PERIOD}}{4} = $ __________

Add the value of the interval size to each x–value.

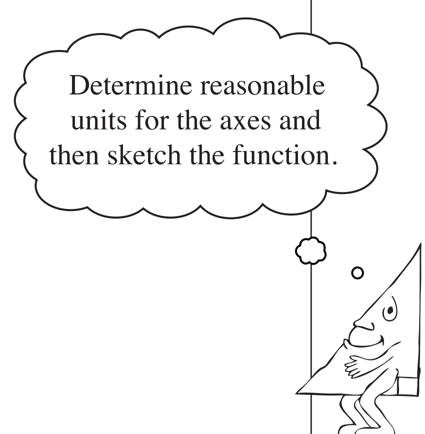
$f(x) = \cos(3x)$					
x	0				
y	1	0	$^{-}1$	0	1



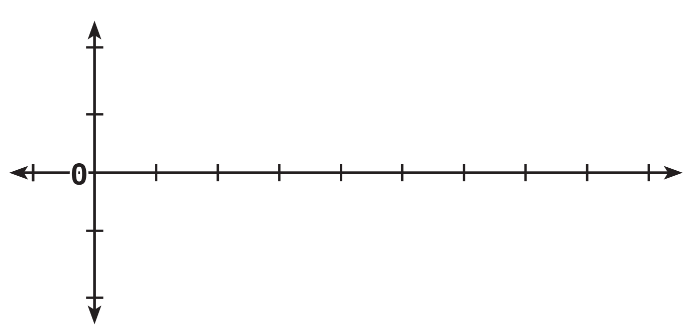
2) $f(x) = \cos(\frac{1}{3}x)$

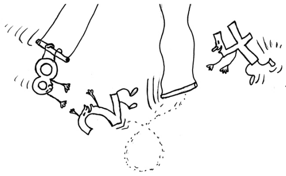
$\text{PERIOD} = \dfrac{2\pi}{b} = $ __________

$\text{INTERVAL SIZE} = \dfrac{\text{PERIOD}}{4} = $ __________



$f(x) = \cos(\frac{1}{3}x)$					
x					
y	1	0	$^{-}1$	0	1





• Amplitude and Period •

Direct Teaching

$f(x) = a\text{SIN}(bx)$

Given the general form of a trigonometric function, the value of ***a*** will determine the amplitude, and stretch or compress the points on the curve vertically. The value of ***b*** determines the period, and will stretch or compress the points horizontally.

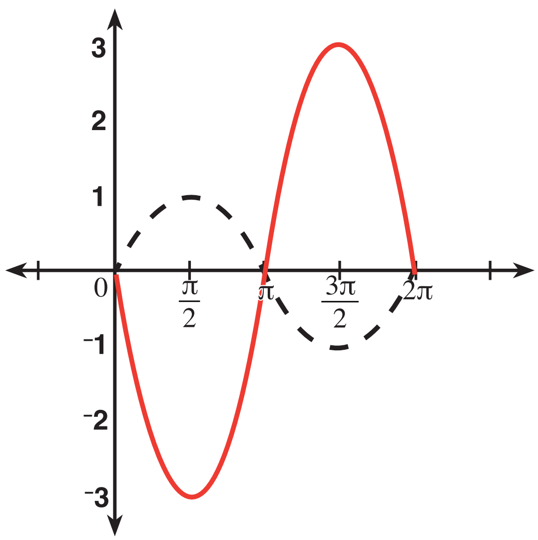
***Try this*:** **Sketch the function.**

1) $f(x) = {}^{-}3\text{SIN}(2x)$

STEP 1: AMPLITUDE = $|a|$ = ______________

Here, the amplitude increases the function by a factor of 3 and reflects it over the x–axis. Fill in the table below, and label the points on the graph.

$f(x) = {}^{-}3\sin(x)$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y					

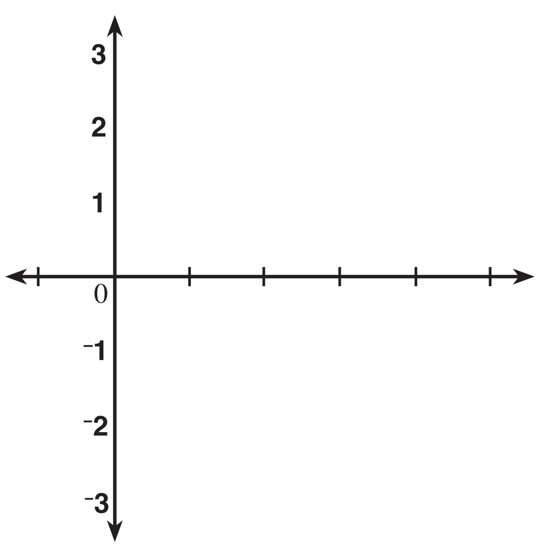


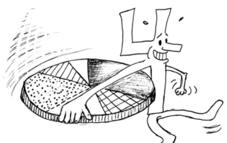
STEP 2: PERIOD = $\frac{2\pi}{b}$ = _________

Find the value of the interval size between the key points and keep the y–values the same.

$\frac{\text{PERIOD}}{4}$ = _________

$f(x) = {}^{-}3\sin(2x)$					
x					
y					





• Amplitude and Period •

☆

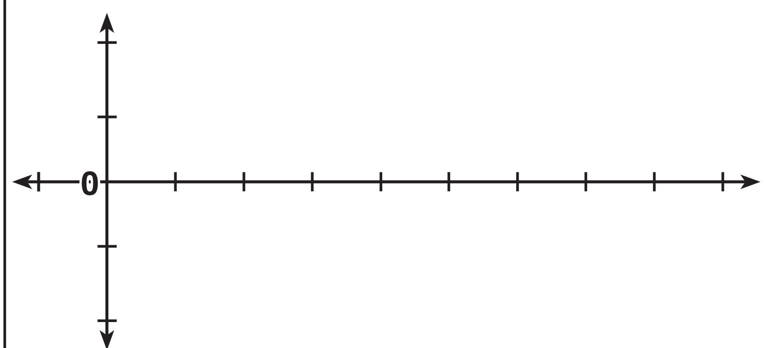
Circle the correct words and sketch the function.

1) $f(x) = \frac{1}{2}\cos(\frac{1}{3}x)$

STEP 1: AMPLITUDE = $|a|$ = ____________

Here, the function's amplitude INCREASES / DECREASES, and the function IS / IS NOT reflected over the ***x***–axis. Fill in the table below, and label the points on the graph.

$f(x) = \frac{1}{2}\cos(x)$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y					



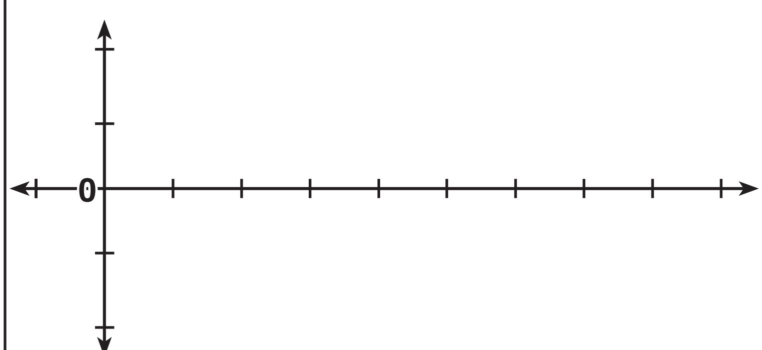
STEP 2: PERIOD = $\frac{2\pi}{b}$ = ________

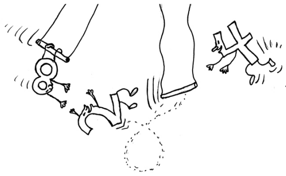
Our ***y***–values are unchanged from the previous step. The period INCREASES / DECREASES.

Divide the period by 4 to find the intervals between the key points.

$\frac{\text{PERIOD}}{4}$ = ________

$f(x) = \frac{1}{2}\cos(\frac{1}{3}x)$					
x					
y					





• Phase Shift •

$$f(x) = \text{SIN}(x - c)$$

The key points of a trigonometric function can shift horizontally.

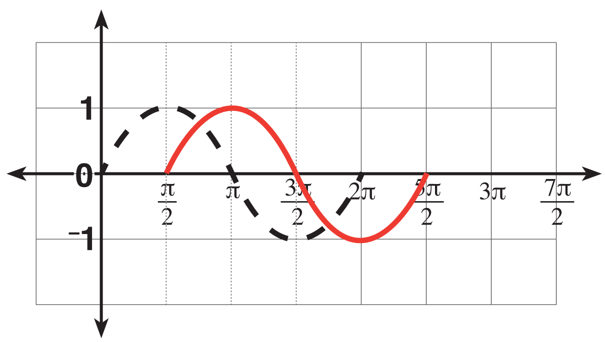
Example: Sketch the function, $f(x) = \text{SIN}(x - \frac{\pi}{2})$.

When the argument in a trigonometric function has a constant added or subtracted, the key points will shift horizontally. Here, the value of $c = \frac{\pi}{2}$. Let's look at the points that make up the function $f(x) = \text{SIN}(x - \frac{\pi}{2})$ and compare them to the points of the original sine function.

$f(x) = \text{SIN}(x)$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	1	0	$^-1$	0

$+\frac{\pi}{2}$

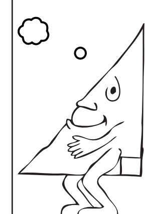
$f(x) = \text{SIN}(x - \frac{\pi}{2})$					
x	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
y	0	1	0	$^-1$	0



The y–values are the same in both cases, but the x–values have the value of c, or $\frac{\pi}{2}$ in this case, added to them. When the value of c is positive, the points are shifted to the right c units. Similarly, if c is negative, the points are shifted to the left c units.

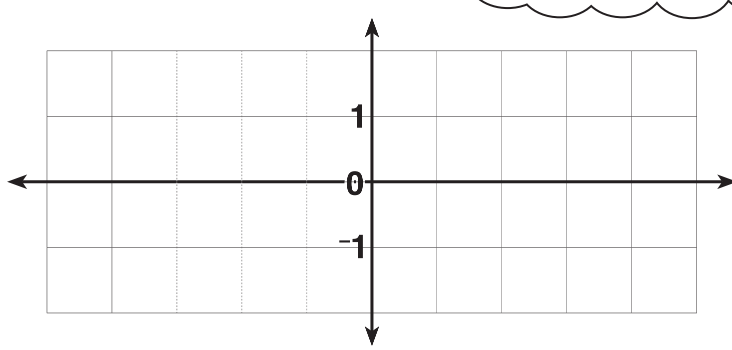
Try this: **Sketch the function.**

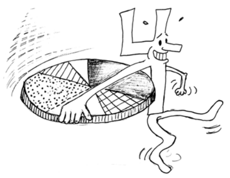
The terms that affect the argument do the opposite of what is expected. This phase shift moves the cosine function to the left.



1) $f(x) = \cos(x + \frac{3\pi}{2})$

$f(x) = \cos(x + \frac{3\pi}{2})$					
x					
y	1	0	$^-1$	0	1





• Period and Phase Shift •

$$f(x) = \sin(bx - c)$$

Given the general form of a trigonometric function, the value of b determines the period and will stretch or compress the points horizontally. The value of c determines the PHASE SHIFT and will shift the points horizontally. Let's look at a general method to determine the transformed key points of a trigonometric function.

Try this: **Sketch the function.**

1) $f(x) = \sin(2x - \pi)$

STEP 1: First, set $(bx - c)$ equal to both 0 and 2π. Setting $(bx - c)$ equal to these values gives us the new starting and ending points of a period compared to the same points of the trigonometric function on its own.

$(bx - c) = (2x - \pi) = 0$ $\qquad$ $(bx - c) = (2x - \pi) = 2\pi$

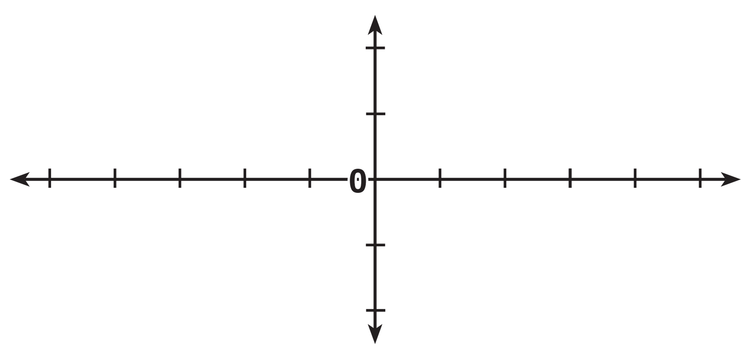
$x =$ ________ $\qquad$ $x =$ ________

STEP 2: Fill in the starting and ending points we found in the step above. We can now find the interval size and divide this value to each of the x–coordinates from the starting point. Make sure to check that you get the same endpoint. The y–values remain unchanged.

$\frac{\text{PERIOD}}{4} =$ ________

$f(x) = \sin(2x - \pi)$					
x					
y	0	1	0	$^{-}1$	0

STEP 3: Plot points and graph the function.



• Period and Phase Shift •

Sketch the function.

1) $f(x) = \cos(\frac{1}{2}x + \pi)$

$(bx - c) = (\frac{1}{2}x + \pi) = 0$

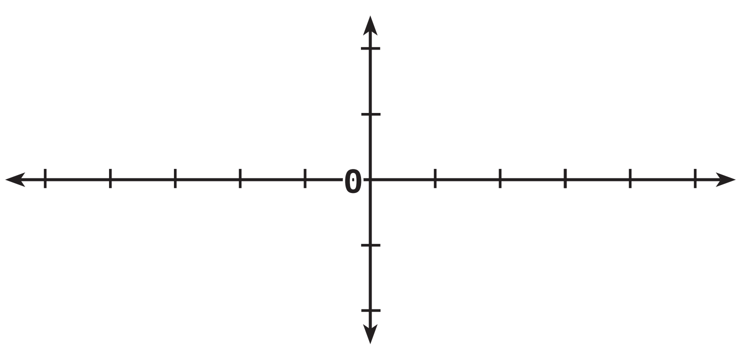
$x =$ ____________

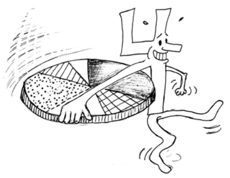
$(bx - c) = (\frac{1}{2}x + \pi) = 2\pi$

$x =$ ____________

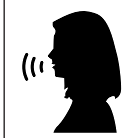
$\frac{\text{PERIOD}}{4} =$ ___________

$f(x) = \cos(2x - \pi)$					
x					
y	1	0	$^{-}1$	0	1





• Vertical Shift •



Direct Teaching

$f(x) = \cos(x) + d$

The key points of a trigonometric function can shift vertically.

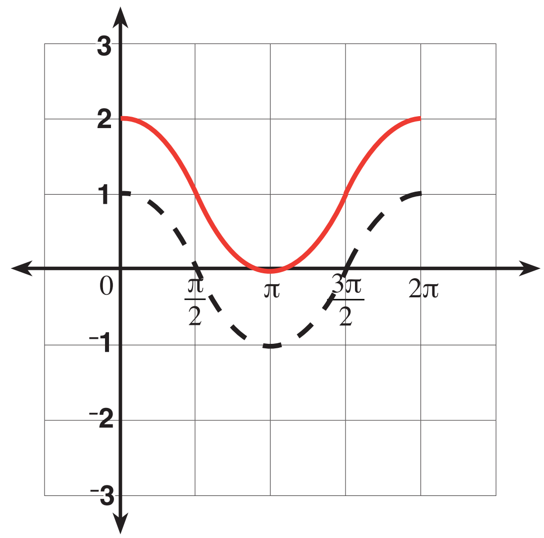
Example: Sketch the function $f(x) = \cos(x) + 1$.

When the trigonometric function has a constant added or subtracted, the key points will shift vertically. Here, the value of $d = 1$. Let's look at the points that make up the function $f(x) = \cos(x) + 1$ and compare them to the points of the original cosine function.

$f(x) = \cos(x)$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	$^{-}1$	0	1

+1

$f(x) = \cos(x) + 1$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	2	1	0	1	2

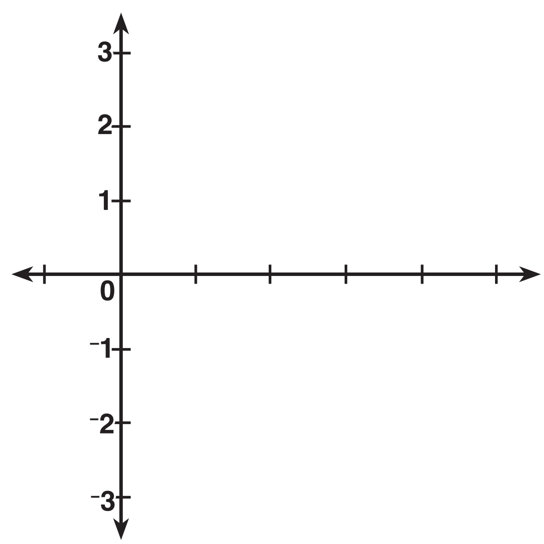


The x–values are the same in both cases, but the y–values have the value of d, 1 in this case, added to them. When the value of d is positive, the points are shifted up by d units. Similarly, if d is negative, the points are shifted down by d units.

Try this:

1) $f(x) = \sin(x) - 1$

$f(x) = \sin(x) - 1$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y					



Sketching Sine and Cosine Functions •

Direct Teaching

$$f(x) = a\sin(bx - c) + d$$

Let's determine the points that make up a single period of a trigonometric function using a step–by–step process. These points can be used to sketch the function.

***Try this*: Determine the points that make up the function.**

1) $f(x) = 2\sin(3x - \frac{3\pi}{2}) - 2$

Step 1: First, determine the AMPLITUDE. There is not a reflection. Multiply the original function's y–values by the amplitude.

AMPLITUDE $= |a| =$ ______

$f(x) = 2\sin x$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y					

Step 2: Set $(bx - c)$ equal to 0 and 2π to determine the starting and ending points of the period. The y–values are not changed from the previous step.

$(3x - \frac{3\pi}{2}) = 0$ $(3x - \frac{3\pi}{2}) = 2\pi$

$x =$ __________ $x =$ __________

$f(x) = 2\sin(3x - \frac{3\pi}{2})$					
x					
y					

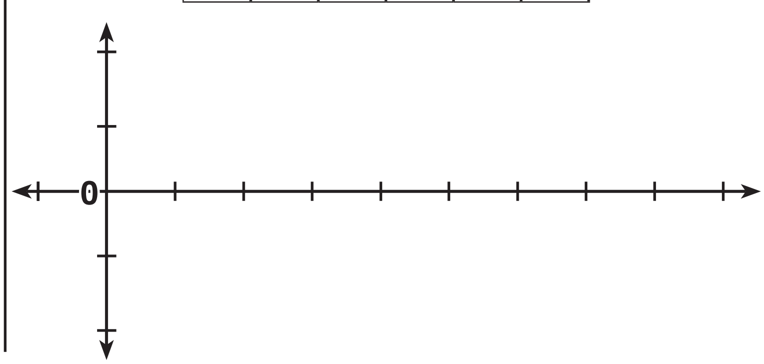
Step 3: We can now find the interval size. We can add this value to each of the x–coordinates from the starting point. The y–values are not changed from the previous step.

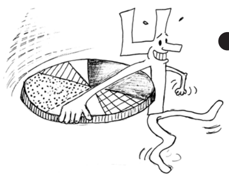
$\frac{\text{PERIOD}}{4} =$ ________

$f(x) = 2\sin(3x - \frac{3\pi}{2})$					
x					
y					

Step 4: Adjust y–values for a vertical shift and sketch the points. The x–values are not changed from the previous step.

$f(x) = 2\sin(3x - \frac{3\pi}{2}) - 2$					
x					
y					





• Sketching Sine and Cosine Functions • ☆

Sketch the function.

1) $f(x) = {}^{-}\text{SIN}(5x + 2\pi) + 1$

STEP 1: First, determine the AMPLITUDE and if there is a reflection. Multiply the original functions *y*–values by the amplitude.

AMPLITUDE $= |a| =$ ______

$f(x) = {}^{-}\text{SIN}\,x$					
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y					

STEP 2: Set $(bx - c)$ equal to 0 and 2π to determine the starting and ending points of the period. The *y*–values are not changed from the previous step.

$(5x + 2\pi) = 0$ $(5x + 2\pi) = 2\pi$

$x =$ __________ $x =$ __________

$f(x) = {}^{-}\text{SIN}(5x + 2\pi)$					
x					
y					

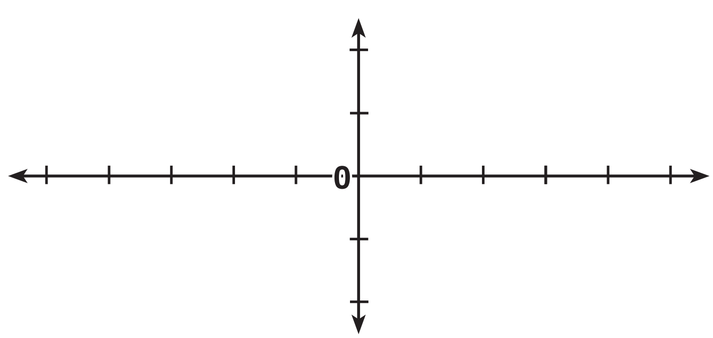
STEP 3: We can now find the interval size. We can add this value to each of the *x*–coordinates from the starting point. The *y*–values are not changed from the previous step.

$\frac{\text{PERIOD}}{4} =$ __________

$f(x) = {}^{-}\text{SIN}(5x + 2\pi)$					
x					
y					

STEP 4: Adjust *y*–values for a vertical shift and sketch the points. The *x*–values are not changed from the previous step.

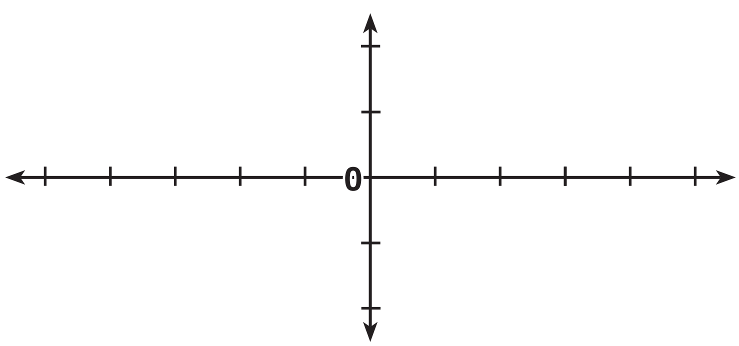
$f(x) = {}^{-}\text{SIN}(5x + 2\pi) + 1$					
x					
y					



• Sketching Sine and Cosine Functions •

Sketch the function.

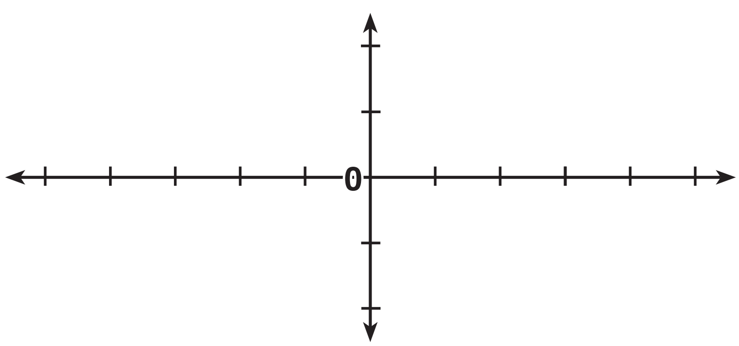
1) $f(x) = \frac{1}{2}\cos(2x + \pi)$

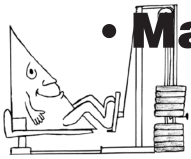


• Sketching Sine and Cosine Functions •

Sketch the function.

1) $f(x) = \frac{1}{4}\sin(x + \pi)$

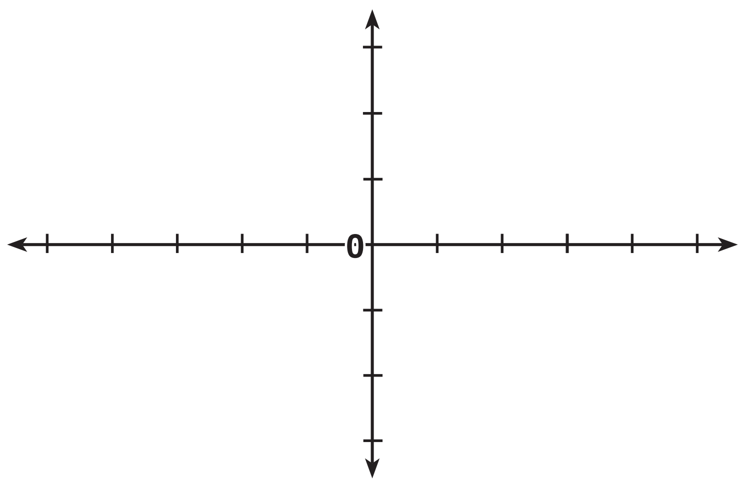




• Mastery Check: Sketching Sine and Cosine Functions •

Sketch the function.

1) $f(x) = 4\sin(x - \frac{\pi}{2}) - 2$



Challenge:

2) $f(x) = \csc(x)$